

Extended colorings for $BSTS(2v + 1)$

M. Gionfriddo*

E. Guardo*

L. Milazzo*

08 June 2011

In memory of Lucia Gionfriddo

Abstract

A $BSTS(2v + 1)$ containing a colorable subsystem $BSTS(v)$ with h -coloring \mathcal{C}' has an extended h -coloring of \mathcal{C}' if it is also h -colorable with a coloring \mathcal{C} in which the subsystem $BSTS(v)$ is colored with \mathcal{C}' . In this paper we give both necessary conditions and sufficient conditions for the existence of an extended coloring. The existence of these colorings is studied either for systems of type $BSTSs(2v + 1)$ containing subsystems $BSTSs(v)$ with $2v + 1 < 103$ or systems of type $BSTSs(2^h + 1)$.

1 Introduction

A triple system $STS(v)$ is defined by a pair (X, \mathcal{B}) , where X is a finite set of vertices with $|X| = v$, and \mathcal{B} is a family of subsets of X , called *blocks*, such that each block contains only three vertices, and any two distinct vertices of X appear together in precisely only one block in \mathcal{B} . It is well known that it must be $v \equiv 1 \text{ or } 3 \pmod{6}$ (see [5]).

A proper k -coloring of (X, \mathcal{B}) is a mapping ϕ from X into a finite set C of k colors, $\{1, 2, \dots, k\}$. In 1993, in [15, 16], V. Voloshin introduced a new coloring for a mixed hypergraph. A mixed hypergraph is a triple $H = (X; \mathcal{C}; \mathcal{D})$, where X is a finite set of vertices, while \mathcal{C} and \mathcal{D} are two families of subsets of X . The elements of \mathcal{C} and \mathcal{D} are called \mathcal{C} -edges and \mathcal{D} -edges respectively. If $\mathcal{C} = \emptyset$, then \mathcal{H} is called a \mathcal{D} -hypergraph, while if $\mathcal{D} = \emptyset$ then \mathcal{H} is called a \mathcal{C} -hypergraph. A strict k -coloring of \mathcal{H} is a vertex coloring

* Department of Mathematics and Informatics, University of Catania, Viale A. Doria, 6 95125 - Catania, Italy. E-mail: gionfriddo@dmf.unict.it, guardo@dmf.unict.it, milazzo@dmf.unict.it.

where any \mathcal{C} -edge has at least two vertices of the same color and any \mathcal{D} -edge has at least two vertices colored differently, and exactly k colors are used in it. If it is not necessary to know the number of used colors then a strict k -coloring will be called strict coloring.

The minimum k for which there exists a strict k -coloring is called the lower chromatic number of \mathcal{H} and is denoted by χ . The strict coloring allow us to characterize the maximum number k for which there exists a strict k -coloring. It is called the upper chromatic number of \mathcal{H} and is denoted by $\overline{\chi}$. If there exists no strict coloring of \mathcal{H} , then \mathcal{H} is said to be uncolorable.

In a \mathcal{D} -hypergraph, the lower chromatic number coincides with the (weak) chromatic number (see [1, 3]) and the upper chromatic number trivially equals v . In a \mathcal{C} -hypergraph, the lower chromatic number trivially equals 1 but the upper chromatic number represents a value that is hard to determine. Mixed hypergraphs with $\mathcal{D} = \mathcal{C}$ are called bi-hypergraphs, and the subsets of X in consideration are called bi-edges. In any proper coloring of bi-hypergraphs, each bi-edge is neither monochromatic (because it is a \mathcal{D} -edge) nor polychromatic (because it is a \mathcal{C} -edge).

In [9], the authors determine hypergraphs that are uncolorable; this problem of uncolorability did not originally exist in the theory of hypergraph coloring (see [1]), as it arises only because of the interaction between \mathcal{D} -edges and \mathcal{C} -edges when a proper coloring is sought and it is called *strong interaction*. For k -colorable hypergraph \mathcal{H} , let r_k be the number of partitions of the vertex set into k nonempty parts (color classes) such that the coloring satisfies the coloring of each \mathcal{C} -edge and each \mathcal{D} -edge. In this case r_k coincides with the number of strict k -colorings if we do not count permutations of the colors.

The integer vector

$$R(\mathcal{H}) = (r_1, \dots, r_k),$$

is called *chromatic spectrum*. The chromatic spectrum, introduced in [15], can be broken (it may have gaps), i.e., it may happen that $r_i = 0$ for some $\chi < i < \overline{\chi}$. The existence of a gap is due to *weak interactions* between \mathcal{C} -edges and \mathcal{D} -edges. The problem of identifying an $STS(v)$ with broken spectrum has not been determined and the only known designs with gaps in the chromatic spectrum are the BP_3 -designs which were found in [4].

In [12], the authors determine two types of coloring of $STS(v)$: 1) $CSTSs(v)$ colorings where each block is a \mathcal{C} -edge and the vertices of each single block can be colored with one or two colors; 2) $BSTSs(v)$ colorings where each block is both a \mathcal{C} -edge and a \mathcal{B} -edge, and the vertices of a block can be

colored only with two colors.

The following two theorems are very important in $CSTSs(v)$ and $BSTSs(v)$ coloring's theory, since they determine the best upper bound of its upper chromatic number, a lower bound for the cardinality of the color classes, and necessary conditions for $BSTS$ colorings.

Theorem 1.1 ([12]) *If S is a $CSTS(v)$ or a $BSTS(v)$ of order $v \leq 2^h - 1$ ($h \in \mathbb{N}$), then $\bar{\chi}(S) \leq h$. \square*

Theorem 1.2 ([14]) *If \mathcal{C} is a strict coloring of a $BSTS$ or $CSTS$ using h colors, then $n_i \geq 2^{i-1}$ for all $1 \leq i \leq h$. \square*

Theorem 1.3 ([10]) *If $S = BSTS(v)$ is colorable with a k -coloring, then the following conditions are true*

$$s_i(s_i - 1) \leq 3 \sum_{j=1}^i n_j(n_j - 1); \quad (i)$$

$$s_k(s_k - 1) = 3 \sum_{j=1}^k n_j(n_j - 1), \quad (k)$$

for every and $1 < i \leq k$. \square

The systems $STSs(2v+1)$ that are obtained by a *doubling plus one construction* are fundamental in the $CSTSs(v)$ and $BSTSs(v)$ coloring's theory. A system $STS(2v+1)$, (X, \mathcal{B}) is obtained from a $STS(v)$, (X', \mathcal{B}') by a doubling plus one construction, if each element of X' can be represented by a symbol x_i for $1 \leq i \leq v$ and if there exists an other set of vertices X'' such that $|X''| = v+1$ and $X' \cap X'' = \emptyset$, and $\mathcal{F} = (F_1, F_2, \dots, F_v)$ is a factorization of the vertices of X'' . The set \mathcal{B} contains all the blocks of \mathcal{B}' and also the blocks of type $\{x_i, y_l, y_m\}$ where $x_i \in X'$ and $(y_l, y_m) \in F_i$. If we consider the set of vertices $X = X' \cup X''$, then the pair (X, \mathcal{B}) defines a $STS(2v+1)$ since $|\mathcal{B}| = \frac{2v(2v+1)}{6}$ and any pair of vertices of X is in one and only one block of \mathcal{B} . The following theorem shows that the unique $CSTSs(v)$ and $BSTSs(v)$ with $\bar{\chi} = h$ and $v \leq 2^h - 1$ are obtained from $STS(3)$ by repeated applications of doubling plus one constructions.

Theorem 1.4 ([12]) *If S is a $CSTS(v)$ or a $BSTS(v)$ with $v \leq 2^h - 1$ and $\bar{\chi} = h$ then:*

1. $v = 2^h - 1$;
2. *in any strict coloring of S with h colors, the color classes have cardinalities*

$$2^0, 2^1, 2^2, \dots, 2^{h-1},$$

and all of them are stable sets;

3. *S is obtained from the $STS(3)$ by repeated applications of doubling plus one constructions.* □

In this paper we study particular colorings of $BSTSs(v)$ obtained by doubling plus one construction, where each block can be colored with only two colors.

Let $S = BSTS(2v+1)$, (X, \mathcal{B}) be obtained by a doubling plus one construction from $S' = BSTS(v)$, (X', \mathcal{B}') which is h -colorable with the coloring \mathcal{C}' . We say that the system $S = BSTS(2v+1)$ has an *h -extended coloring* of \mathcal{C}' if there exists a h -coloring \mathcal{C} of S such that the subsystem S' is colored with \mathcal{C}' . This is equivalent to say that the h -coloring \mathcal{C} of S depends only on the coloring of the vertices of $X'' = X - X'$ by the colors of \mathcal{C}' . In [7] the authors proved that extended h -colorings don't exist if $v = 2^h - 1$ and $h < 10$, and it leaves as an open problem the case where $h \geq 10$; in particular, using theorems 1.1 and 1.4, this last result allows us to get information on the lower and upper chromatic numbers.

Theorem 1.5 ([7]) *If a $BSTS(2^h - 1)$ or a $CSTS(2^h - 1)$ is obtained by a sequence of doubling constructions plus one starting from $STS(3)$, then $\bar{\chi} = \chi = h$ for all $h < 10$.* □

Our paper is structured as follows. In Section 2 we give necessary conditions for the existence of extended h -colorings for generic systems $BSTSs(2v+1)$. We also give a sufficient condition to determine particular extended colorings. Section 3 is devoted to determine which systems $BSTSs(2v+1)$, with $2v+1 < 103$, obtained from $BSTS(v)$ by doubling plus one construction, have extended colorings. In Section 4 we give necessary conditions for the

existence of extended colorings of particular systems $BSTS(2^h - 1)$. For each of them, we classify the cardinalities of the color classes of possible extended colorings and for every $h \geq 1$, we also determine an infinite class of $BSTS(2^h - 1)$ that cannot be colored by extended colorings.

2 Extended colorings

Let $S' = BSTS(v)$, (X', \mathcal{B}') be a h -colorable system with a coloring $\mathcal{C}' = \{n'_1, n'_2, \dots, n'_h\}$ and let $S = BSTS(2v + 1)$, (X, \mathcal{B}) be a system obtained by doubling plus one construction plus one from S' . The system S is certainly $(h + 1)$ -colorable with a coloring $\mathcal{C}'' = \{n''_1, n''_2, \dots, n''_h, n''_{h+1}\}$ where $n''_{h+1} = v + 1$. This coloring is obtained coloring the subsystem S' with \mathcal{C}' and the vertices in $X'' = X - X'$, where $|X''| = v + 1$, are colored using a new color, denoted by $[h + 1]$, different from the h colors in \mathcal{C}' . We are considering the problem to color the system S with an extended h -coloring $\mathcal{C} = \{n_1, n_2, \dots, n_h\}$ where the vertices of the subsystem S' are colored with \mathcal{C}' . This extended coloring depends only on the colors assigned to the vertices in X'' , and on the fact that they are colored with just the colors used in \mathcal{C}' .

The values $c_i = n_i - n'_i$, with $1 \leq i \leq h$, are the numbers of vertices in X'' which are colored with the color $[i] \in \mathcal{C}'$. It is possible that $c_j = 0$, for some $1 \leq j \leq h$, and it is evident that $\sum_{i=1}^h c_i = v + 1$. A necessary condition for the existence of an extended h -coloring of \mathcal{C}' for S is given by following theorem.

Theorem 2.1 *Let $S = BSTS(2v + 1)$, (X, \mathcal{B}) be a system obtained by doubling plus one construction from $S' = BSTS(v)$, (X', \mathcal{B}') h -colorable with the coloring $\mathcal{C}' = \{n'_1, n'_2, \dots, n'_h\}$. If $\mathcal{C} = \{n_1, n_2, \dots, n_h\}$ is an extended h -coloring of \mathcal{C}' for S where $n_i = n'_i + c_i$, then the two following equations are both satisfied:*

$$\begin{cases} \sum_{i=1}^h c_i^2 + 2 \sum_{i=1}^h n'_i c_i = (v + 1)^2 \\ \sum_{i=1}^h c_i = v + 1. \end{cases} \quad (1)$$

Proof.

Since we know that $\sum_{i=1}^h c_i = v + 1$, we need to prove only the first equality of (1). Let \mathcal{F} be the factorization of $X'' = X - X'$ defined in the doubling plus one construction from which we get S , and let $F_l \in \mathcal{F}$ be one of its factor corresponding to the vertex $x_l \in X'$ colored with a color $[i]$.

In the factorization \mathcal{F} , the numbers of different monochromatic and non monochromatic pairs are $\sum_{i=1}^h \binom{c_i}{2}$ and $\sum_{i=1}^h n_i c_i$, respectively. A generic monochromatic pair colored with the color $[j]$ has to be in a factor corresponding to the vertices $x_l \in X'$ colored with $[k]$, where $[j] \neq [k]$. Since in a doubling plus one construction any two vertices of X'' are in only one block which contains a vertex of X' , then a generic non monochromatic pair colored with $[j]$ and $[k]$ has to be in a factor corresponding to a vertex $x_l \in X'$ colored either with the color $[j]$ or $[k]$. If we consider $c_j \neq 0$, for some j , then in all of the factors corresponding to the vertices $x_l \in X'$ of color $[j]$ there are exactly c_j non monochromatic pairs.

In \mathcal{F} , the number of all pairs is $\frac{v(v+1)}{2}$ and, therefore, we have that

$$\sum_{i=1}^h \binom{c_i}{2} + \sum_{i=1}^h n'_i c_i = \frac{v(v+1)}{2}.$$

It easy to check, by simple calculation, that we obtain the first equality of (1), and hence, the theorem follows. \square

The solutions of the system (1) given in theorem 2.1 are called *solutions with respect to \mathcal{C}'* . It is important to note that this theorem gives only necessary conditions for the existence of an extended coloring; indeed, in [7] the authors determine solutions of the system (1) for $BSTS_s(2^h - 1)$, with $6 \leq h < 10$, but these solutions do not define any extended coloring.

The following corollary gives other conditions useful to find solutions for the system (1) which could define extended colorings.

Corollary 2.1 *Let $S = BSTS(2v+1)$ be a system obtained by doubling plus one construction from a h -colorable $S' = BSTS(v)$ with the coloring $\mathcal{C}' = \{n'_1, n'_2, \dots, n'_h\}$, and let $\{c_1, c_2, \dots, c_h\}$ be a solution with respect to \mathcal{C}' .*

1. *If there exists a $c_j = 0$ then all the $c_i > 0$ are even.*
2. *If there exists a $c_j > \frac{v+1}{2}$ then it does not exist an extended coloring of \mathcal{C}' .*

Proof.

1. If there exists a $c_i = 0$, then in all of the factors corresponding to the vertices $x_l \in X'$ colored with $[i]$ there are all the monochromatic pairs, so every $c_k > 0$ has to be even.
2. if there exists a $c_j > \frac{v+1}{2}$, then in all of the factors corresponding to the vertices $x_l \in X'$ colored with $[j]$ there are monochromatic pairs of color $[j]$,

and this is not possible. \square

The first $BSTSs(v)$ which can be obtained by doubling plus one construction are $BSTSs(7)$ and $BSTSs(15)$. But, as it was proved in [7], these systems have not solutions with respect to $\mathcal{C}' = \{1, 2\}$ and to $\mathcal{C}'' = \{1, 2, 2^2\}$, so they have not extended colorings. The next systems are $BSTSs(19)$ obtained by doubling plus one construction from $BSTS(9)$. This latter system can be colored with an unique 3-coloring $\mathcal{C}' = \{1, 4, 4\}$ (see [13]). All the solutions with respect to \mathcal{C}' are in the table 1.

c_1	c_2	c_3
3	2	5
3	5	2
5	0	5
5	5	0
8	0	2
8	2	0

table 1

From corollary 2.1, all the solutions of table 1, with the exception of the first and the second ones, do not permit to obtain extended colorings. In the first solution, since $c_1 = 3$ and $c_3 = 5$, it is necessary that in the four factors corresponding to the vertices $x_l \in X'$ colored with [2], there are exactly two non monochromatic pairs colored with [1] and [2] and with [2] and [3], one monochromatic pair of color [1] and two monochromatic pairs of color [3]. Because $c_1 = 3$ we have that these dispositions are not permitted, so this solution does not permit to define an extended coloring. Analogously, it is possible to prove that also the second solution of table 1 does not permit an extended coloring, therefore there are not extended colorings for $BSTSs(19)$. In [10] it was proved that there are $BSTSs(19)$ uniquely 3-colorable, uniquely 4-colorable and 3 and 4-colorable, so we can consider the following proposition.

Proposition 2.1 *All the $BSTSs(19)$, which are 3 and 4-colorable or uniquely 3-colorable, do not contain a subsystem $BSTS(9)$.* \square

The following theorem gives a sufficient condition for the existence of extended h -colorings of $BSTS(2v+1)$ when there is a h -colorings \mathcal{C}' of system $BSTS(v)$, and it permits to us to find extended colorings without resolving system (1).

Theorem 2.2 *Let $S' = BSTS(v)$, (X', \mathcal{B}') be a system h -colorable with $\mathcal{C}' = (n'_1, n'_2, \dots, n'_h)$. If there exist p integers n'_{k_i} , with $1 \leq i \leq p$ and $h > p$, where $n'_{k_1} + n'_{k_2} = (v+1)/2^{p-1}$ is an even integer, and $n'_{k_i} = (v+1)/2^{p-i+1}$, for $3 \leq i \leq p$, are all even, then $S = BSTS(2v+1)$, (X, \mathcal{B}) , obtained by doubling construction plus one from S' , has an extended h -coloring of \mathcal{C}' .*

Proof.

Set $c_{k_1} = (v+1)/2^{p-1}$, $c_{k_2} = (v+1)/2^{p-1}$, $c_{k_i} = (v+1)/2^{p-i+1}$ for $3 \leq i \leq p$, and $c_j = 0$ for all $j \neq k_i$ with $1 \leq i \leq p$. It is necessary that these assignments give correct numbers of monochromatic and non monochromatic pairs in a factorization \mathcal{F} of X'' which defines a coloring of $BSTS(2v+1)$, i.e., it is necessary to check if (c_1, c_2, \dots, c_h) is a solution with respect to \mathcal{C}' of the system

$$\begin{cases} \sum_{i=1}^p c_{k_i}^2 + 2 \sum_{i=1}^p n'_{k_i} c_{k_i} = (v+1)^2 \\ \sum_{i=1}^p c_{k_i} = (v+1). \end{cases}$$

Replacing the values of c_{k_i} and n'_{k_i} , for $1 \leq i \leq p$, the first equality becomes

$$\sum_{i=1}^{2p-2} \frac{(v+1)^2}{2^i} + \frac{(v+1)^2}{2^{2p-2}} = (v+1)^2,$$

and it is trivially true.

Set $S = \sum_{i=2}^p (v+1)/2^{p-i+1} + (v+1)/2^{p-1}$. It is simple to verify that the difference $S = 2S - S$ is equal to $v+1$, so also the second equality is true. In this second part of the proof, we are going to specify the modalities of the distribution of the vertices in X'' and of the colors $[k_i]$, with $1 \leq i \leq h$, in a factorization \mathcal{F} in such a way the rules of an extended coloring of \mathcal{C}' are respected.

Let us color exactly $(v+1)/2$ vertices with the color $[k_p]$ and the others $(v+1)/2$ with the colors $[k_i]$ with $1 \leq i \leq p-1$. Easily, it is possible to build $(v+1)/2$ factors with non monochromatic pairs using for each one of them colors $[k_p]$ and $[k_i]$, with $1 \leq i \leq p-1$. These factors are connected with the $(v+1)/2$ vertices $x_l \in X'$ colored with $[k_p]$. The set of the $(v+1)/2$ vertices in X'' colored with $[k_p]$ defines a factorization $\mathcal{F}^{(1)}$ of $(v-1)/2$ factors all containing $(v+1)/2^2$ monochromatic pairs of color $[k_p]$. The factors in $\mathcal{F}^{(1)}$ are placed on the bottom of the $(v-1)/2$ factors in \mathcal{F} corresponding to the vertices $x_l \in X'$ colored with all the colors distinct from $[k_p]$. Now, let us consider $(v+1)/2^2$ vertices of X'' colored with the color $[k_{p-1}]$; all the non monochromatic pairs that use the colors

$[k_{p-1}]$ and $[k_i]$, with $1 \leq i \leq p-2$, define $(v+1)/2^2$ factors all containing $(v+1)/2^2$ pairs. These last factors cover completely $(v+1)/2^2$ factors of \mathcal{F} which contain $(v+1)/2^2$ monochromatic pairs of color $[k_p]$ of $\mathcal{F}^{(1)}$, and they are connected with the vertices $x_l \in X'$ colored with $[k_{p-1}]$. The $(v+1)/2^2$ vertices of X'' colored with $[k_{p-1}]$ define a factorization $\mathcal{F}^{(2)}$ of $(v+1)/2^2 - 1$ factors. The pairs of these factors are added to the other incomplete factors of \mathcal{F} which contain $(v+1)/2^2$ monochromatic pairs colored with $[k_p]$ and contained in the factors of $\mathcal{F}^{(1)}$. Therefore in these last factors there are $(v+1)/2^2$ pairs of color $[k_p]$ and $(v+1)/2^4$ pairs of color $[k_{p-1}]$. We repeat this procedure until that the $(v+1)/2^{p-2}$ vertices in X'' colored with $[k_3]$ define $(v+1)/2^{p-2}$ factors of non monochromatic pair colored with the colors $[k_3]$ and $[k_i]$, with $i = 1$ and 2 . They completely cover $(v+1)/2^{p-2}$ factors of \mathcal{F} corresponding to the vertices $x_l \in X'$ colored with $[k_3]$. Also the vertices of X'' colored with $[k_3]$ define a factorization $\mathcal{F}^{(p-2)}$ of $(v+1)/2^{p-2} - 1$ factors of $(v+1)/2^{p-1}$ monochromatic pairs colored with $[k_3]$. The factors in $\mathcal{F}^{(p-2)}$ are posed on the remanning $(v+1)/2^{p-2} - 1$ factors of \mathcal{F} which are not complete and containing monochromatic pairs of colors $[k_i]$ for $3 \leq i \leq p$. Finally the $(v+1)/2^{p-1}$ vertices of color $[k_1]$ and the $(v+1)/2^{p-1}$ vertices of colors $[k_2]$ define $(v+1)/2^{p-1}$ containing non monochromatic pairs and colored with $[k_1]$ and $[k_2]$. These factors completely cover all the $(v+1)/2^{p-1}$ factors of \mathcal{F} corresponding with the vertices $x_l \in X'$ colored with $[k_1]$ and $[k_2]$. The vertices of X'' colored with $[k_1]$ and $[k_2]$ define respectively two factorizations $\mathcal{F}^{(p-1)}$ and $\mathcal{F}^{(p)}$ of $(v+1)/2^{p-1} - 1$ factors all containing $(v+1)/2^p$ monochromatic pairs of colors $[k_1]$ and $[k_2]$. These factors cover completely all the remanning $(v+1)/2^{p-1} - 1$ factors of \mathcal{F} corresponding to the vertices $x_l \in X'$ colored with $[j] \neq [k_i]$ with $1 \leq i \leq p$. We obtain a factorization \mathcal{F} of $\sum_{i=1}^{p-1} (v+1)/2^i + (v+1)/2^{p-1} - 1 = v$ factors which gives a correct coloring for the BSTS($2v+1$) obtained by doubling construction plus one and the theorem follows. \square

In the previous theorem it is possible to obtain the factorization \mathcal{F} since all quantities $(v+1)/2^i$ with $1 \leq i \leq p-1$ are even, therefore it is always possible to construct the factorizations $\mathcal{F}^{(i)}$, for $1 \leq i \leq p-1$. Notice that in this theorem the solution with respect to \mathcal{C}' has at least a $c_l = 0$ with $1 \leq l \leq h$, since in \mathcal{F} there are factors with only monochromatic pairs. The following corollary fixes the particular case when $p = 2$.

Corollary 2.2 *Let $S' = \text{BSTS}(v)$ be an h -colorable system with the coloring $\mathcal{C}' = (n_1, n_2, \dots, n_h)$, if there are two n_i and n_j such that $n_i + n_j = \frac{(v+1)}{2}$ is even, then the BSTS($2v+1$) obtained by doubling construction plus one*

from S' has an extended coloring with respect to \mathcal{C}' .

Proof.

We obtain the proof using the same technique of theorem 2.2 assigning the color $[i]$ to $(v+1)/2$ vertices of X'' and the color $[j]$ to the remaining vertices. \square

The following corollary allows us to determine infinite classes of BSTSs having extended colorings.

Corollary 2.3 *Let $S' = \text{BSTS}(v)$, (X', \mathcal{B}') be a system h -colorable with the coloring $\mathcal{C}' = (n'_1, n'_2, \dots, n'_h)$. If there exist p integers n'_{k_i} , with $1 \leq i \leq p$, such that $n'_{k_1} + n'_{k_2} = (v+1)/2^{p-1}$ is even and $n'_{k_i} = (v+1)/2^{p-i+1}$, for $3 \leq i \leq p$, are all even, then all the BSTSs $(2^k \cdot (v+1) - 1)$, with k positive integer and obtained from a sequence of doubling constructions plus one from S' , have an extended $(h+k-1)$ -coloring.*

Proof.

In general every $\text{BSTS}(2^k \cdot (v+1) - 1)$ has an extended $(h+k-1)$ -coloring with respect to the coloring $\mathcal{C}'' = (n''_1, n''_2, \dots, n''_h, n''_{h+1}, \dots, n''_{h+k-1})$ of the system $\text{BSTS}(2^{k-1} \cdot (v+1) - 1)$ where $n''_i = n'_i$, with $1 \leq i \leq h$, and $n''_j = 2^{j-h-1} \cdot (v+1)$ with $h+1 \leq j \leq h+k-1$, therefore by theorem 2.2 the corollary is true. \square

The first following proposition gives a necessary condition for the existence of an extended colorings connected with a solutions of the system (1) with respect to the coloring \mathcal{C}' , while the second one permits to characterize solutions of system (1) which do not give extended colorings.

Proposition 2.2 *Let \mathcal{C} be an extended coloring of $\mathcal{C}' = (n'_1, n'_2, \dots, n'_h)$ for the system $S = \text{BSTS}(2v+1)$, (X, \mathcal{B}) , obtained by a doubling plus one construction from a $S' = \text{BSTS}(v)$, (X', \mathcal{B}') . If in a solution (c_1, c_2, \dots, c_h) with respect to \mathcal{C}' there are two $c_i > 0$ and $c_j > 0$, with $1 \leq i, j \leq h$ and $i \neq j$, then we have that $c_i \leq n'_i + n'_j$ and $c_j \leq n'_i + n'_j$.*

Proof.

Let us consider $x' \in X''$ colored with $[i]$, it is in c_j non monochromatic pairs colored with $[i]$ and $[j]$ all contained in different factors. These factors are corresponding to the vertices $x_l \in X'$ colored only with either $[i]$ or $[j]$ and they are at most $n'_i + n'_j$, so $c_j \leq n'_i + n'_j$. Analogously, we obtain that

$$c_i \leq n'_i + n'_j. \quad \square$$

The previous proposition has the best utility when in \mathcal{C}' there is a $n'_i = 1$ and $c_i > 0$, in fact in this case $c_j \leq +n'_j + 1$ for every $c_j > 0$ with $1 \leq j \leq h$ and $i \neq j$. In the general case, if we want the best evaluation of a solution of system (1) with respect to the conditions of proposition 2.2, then we have to find in \mathcal{C}' the value of n'_i such that $c_i > 0$, $n'_i \leq n'_j$ for every $1 \leq j \leq h$, with $c_j > 0$ and $i \neq j$. This particular choice of n'_i permits to optimize the inequalities $c_j \leq +n'_j + n'_i$ with $i \neq j$.

Proposition 2.3 *Let $S = BSTS(2v+1)$, (X, \mathcal{B}) be a system obtained by doubling plus one construction from the system $S' = BSTS(v)$, (X', \mathcal{B}') , colorable with the coloring $\mathcal{C}' = (n'_1, n'_2, \dots, n'_h)$. If (c_1, c_2, \dots, c_h) is a solution of system (1) with respect to \mathcal{C}' , with $c_l > 0$ for $1 \leq l \leq h$, and $c_i = (v+1)/2$, and with a $c_j > 0$ such that $(\sum_k n'_k) \cdot \lfloor c_j/2 \rfloor < c_j(c_j-1)/2$, with $k \neq i$ and j , then the solution (c_1, c_2, \dots, c_h) does not determine an extended h -coloring \mathcal{C}' .*

Proof.

In X'' , the solution (c_1, c_2, \dots, c_h) defines $c_j(c_j-1)/2$ monochromatic pairs of color $[j]$ which are not in the factors corresponding to the vertices $x_l \in X'$ colored with $[i]$ and $[j]$. These pairs have to be in $\sum_k n'_k$ factors, with $k \neq i, j$ and corresponding to the vertices $x_l \in X'$ of color $[k]$. In these factors there are at most $\lfloor c_j/2 \rfloor$ monochromatic pairs of color $[j]$, therefore if $(\sum_k n'_k) \cdot \lfloor c_j/2 \rfloor < c_j(c_j-1)/2$, then it is not possible to obtain an extended coloring. \square

3 Extended colorings for small BSTSs

In the previous section it was proved that there are not extended colorings for $BSTS(v)$ with $v \leq 19$. In this section we study the first systems with extended colorings.

Consider the system $BSTS(27)$ obtained by doubling plus one construction from $BSTS(13)$ which can be uniquely 3-colorable with $\mathcal{C}' = (2, 5, 6)$ (see [11]). All the solutions of the system (1) with respect to \mathcal{C}' are in the table 2.

c_1	c_2	c_3
4	4	6
7	1	6
4	7	3
7	7	0
10	1	3
10	4	0

table 2

Theorem 3.1 *There exists a BSTS(27), (X, \mathcal{B}) , obtained by doubling plus one construction from one BSTS(13), (X', \mathcal{B}') , which has an extended 3-coloring of $\mathcal{C}' = (2, 5, 6)$. For this system we have that $\chi = 3$ e $\bar{\chi} = 4$.*

Proof.

Table 2 shows all the six solutions of the system (1) with respect to the coloring \mathcal{C}' . From corollary 2.1, the last three solutions of this table do not define extended colorings of \mathcal{C}' .

Consider the solution (4, 7, 3), there are three monochromatic pairs of color [3]. We can put two of them in two factors of \mathcal{F} corresponding to the two vertices $x_l \in X'$ of color [1], but the third one can be neither in the five factors corresponding to the vertices $x_l \in X'$ of color [2] nor in the six factors corresponding to the vertices $x_l \in X'$ of color [3], so this solution cannot define an extended coloring of \mathcal{C}' .

The solutions (4, 4, 6) and (7, 1, 6) define both extended colorings of \mathcal{C}' of type $\mathcal{C} = (6, 9, 12)$. These colorings are, respectively, represented in the appendix by the factorization of table 16 where we have to consider the following coloring classes $X_1 = \{1, 2, 14, 15, 16, 17\}$, $X_2 = \{3, 4, 5, 6, 7, 18, 19, 20, 21\}$ and $X_3 = \{8, 9, 10, 11, 12, 13, 22, 23, 24, 25, 26, 27\}$, and by the factorization of table 17 with coloring classes $X_1 = \{1, 2, 14, 15, 16, 17, 18, 19, 20\}$, $X_2 = \{3, 4, 5, 6, 7, 21\}$ and $X_3 = \{8, 9, 10, 11, 12, 13, 22, 23, 24, 25, 26, 27\}$.

If a BSTS(27) has an extended 3-coloring with respect to \mathcal{C}' then it is also 4-colorable with the coloring $\mathcal{C}'' = (2, 5, 6, 14)$. Suppose that there exists a 5-coloring for a BSTS(27), by theorems 1.1 and 1.2 for the five coloring classes we have that $n_i \geq 2^{i-1}$ for $1 \leq i \leq 5$. But $\sum_{i=1}^5 n_i > 27$, and this is not possible. Therefore every colorable BSTS(27) cannot be 5-colorable, and a BSTS(27) with an extended 3-coloring has $\chi = 3$ and $\bar{\chi} = 4$. \square

The BSTS(27) is the smallest BSTS which can have an extended coloring. The next two BSTSs obtained by doubling plus one construction are BSTSs(31) and BSTSs(39).

In [7], it was proved that BSTSs(31) have not extended colorings of $\mathcal{C}' = (1, 2, 2^2, 2^3)$. The BSTSs(39) obtained by doubling plus one construction from a BSTS(19) can have extended coloring. In [10] it was proved that a BSTS(19) can be colored just with the following colorings $\mathcal{C}'_1 = (4, 6, 9)$, $\mathcal{C}'_2 = (1, 2, 8, 8)$ and $\mathcal{C}'_3 = (1, 4, 4, 10)$.

Theorem 3.2 *There are colorable BSTSs(39), obtained by doubling plus one construction from a BSTS(19), with extended coloring of $\mathcal{C}'_1 = (4, 6, 9)$ and $\mathcal{C}'_2 = (1, 2, 8, 8)$. In particular for these systems there are BSTSs(39) with $\chi = 3$ and $\bar{\chi} = 4$, with $\chi = 4$ and $\bar{\chi} = 5$ and with $\chi = 3$ and $\bar{\chi} = 5$.*

Proof.

In [10] it was proved that there exist BSTSs(19) uniquely 3-colorable with the coloring $\mathcal{C}'_1 = (4, 6, 9)$, BSTSs(19) uniquely 4-colorable with the coloring $\mathcal{C}'_2 = (1, 2, 8, 8)$ and BSTSs(19) 3 and 4-colorable with the colorings $\mathcal{C}'_1 = (4, 6, 9)$ and $\mathcal{C}'_2 = (1, 2, 8, 8)$, then we have that $\bar{\chi} < 5$ for the colorable BSTSs(19).

Since in \mathcal{C}'_1 there are two n'_1 and n'_2 with $n'_1 + n'_2 = 4 + 6 = (v + 1)/2$ and in \mathcal{C}'_2 two n'_2 e n'_i , where $i = 3$ or 4 , with $n'_1 + n'_i = 2 + 8 = (v + 1)/2$, then by corollary 2.2 we have that for BSTSs(39) there are extended 3-colorings of \mathcal{C}'_1 and of \mathcal{C}'_2 .

The BSTSs(39), obtained by doubling plus one construction from a uniquely 3-colorable BSTS(19), can be colored with the extended coloring $\mathcal{C}_1 = (9, 14, 16)$ and with the 4-coloring $\mathcal{C}_3 = (4, 6, 9, 20)$. These systems are not 5-colorable because their subsystem BSTS(19) is uniquely 3-colorable, therefore $\chi = 3$ and $\bar{\chi} = 4$.

The BSTSs(39), obtained by doubling plus one construction from a uniquely colorable BSTS(19) with \mathcal{C}'_2 , can be colored with the extended coloring $\mathcal{C}_2 = (1, 12, 8, 18)$ and with the 5-coloring $\mathcal{C}_4 = (1, 2, 8, 8, 20)$. These systems cannot be 3-colorable because their subsystem BSTS(19) is uniquely 4-colorable, and $\chi = 4$ e $\bar{\chi} = 5$.

Finally, the BSTSs(39), obtained by doubling plus one construction from a colorable BSTS(19) either with \mathcal{C}'_1 or with \mathcal{C}'_2 , can be 3, 4 and 5-colorable with the extended coloring $\mathcal{C}_1 = (9, 14, 16)$, with the coloring $\mathcal{C}_3 = (4, 6, 9, 20)$ and with the coloring $\mathcal{C}_4 = (1, 2, 8, 8, 20)$, and in this last case $\chi = 3$ and $\bar{\chi} = 5$. \square

BSTS(43) is the next systems which can be obtained by doubling plus one construction from a BSTS(21). A BSTS(21) can be uniquely 3-colorable with one of the two colorings: $\mathcal{C}'_1 = (5, 6, 10)$ and $\mathcal{C}'_2 = (4, 8, 9)$ (see [10]).

All the solutions of the system (1) with respect to \mathcal{C}'_1 and \mathcal{C}'_2 are in the tables 5 and 6, respectively.

c_1	c_2	c_3	c_1	c_2	c_3
5	10	7	6	8	8
5	11	6	6	9	7
11	4	7	12	2	8
11	11	0	12	9	1
12	4	6	13	2	7
12	10	0	13	8	1
table 5			table 6		

Theorem 3.3 *The BSTSs(43), (X, \mathcal{B}) , obtained by doubling plus one construction from a colorable BSTS(21), (X', \mathcal{B}') either with $\mathcal{C}'_1 = (5, 6, 10)$ or with $\mathcal{C}'_2 = (4, 8, 9)$, have respectively extended colorings. In particular, for these systems it is $\chi = 3$ and $\bar{\chi} = 4$.*

Proof.

Table 5 shows all the solutions with respect to \mathcal{C}'_1 . In this table, from corollary 2.1, the last three solutions cannot define extended colorings and, from proposition 2.3, the third solution does not define extended colorings. The first and the second solutions of table 5 define extended colorings as we can see in the appendix, tables 18 and 19, respectively. In these tables we have the coloring classes $X_1 = \{1, 2, 3, 4, 5, 22, 23, 24, 25, 26\}$, $X_2 = \{6, 7, 8, 9, 10, 11, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36\}$, and $X_3 = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 37, 38, 39, 40, 41, 42, 43\}$ for the coloring connected with the first solution, and the coloring classes $X_1 = \{1, 2, 3, 4, 5, 22, 23, 24, 25, 26\}$, $X_2 = \{6, 7, 8, 9, 10, 11, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37\}$ and $X_3 = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 38, 39, 40, 41, 42, 43\}$ for the coloring connected with the second solution.

In table 6, that defines solutions with respect to \mathcal{C}'_2 , using corollary 2.1 we have that the last four solutions cannot define extended colorings. The first and the second solutions define extended colorings as we can see in tables 20 and 21 of the appendix. The extended coloring of \mathcal{C}'_2 connected to the first solution has as coloring classes the following ones $X_1 = \{1, 2, 3, 4, 22, 23, 24, 25, 26, 27\}$, $X_2 = \{5, 6, 7, 8, 9, 10, 11, 12, 28, 29, 30, 31, 32, 33, 34, 35\}$, $X_3 = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 36, 37, 38, 39, 40, 41, 42, 43\}$, while the coloring classes of the extended coloring connected to the second solution are $X_1 = \{1, 2, 3, 4, 22, 23, 24, 25, 26, 27\}$, $X_2 = \{5, 6, 7, 8, 9, 10, 11, 12, 28, 29, 30, 31, 32, 33, 34, 35, 36\}$, $X_3 = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 37, 38, 39, 40, 41, 42, 43\}$.

In [8] it was proved that for BSTSs(43), it is $\bar{\chi} \leq 4$, and hence, the theorem is proved. \square

The systems BSTS(51), obtained from doubling plus one construction from a BSTS(25), have not extended colorings of the unique colorings $\mathcal{C}'_1 = (5, 10, 10)$ and $\mathcal{C}'_2 = (1, 4, 8, 12)$ of colorable BSTSs(25) (see [2, 8]).

c_1	c_2	c_3	c_4
0	0	0	26
0	0	6	20
0	16	6	4
0	22	0	4
1	0	8	17
1	13	8	4
3	13	8	2
3	16	6	1
3	22	0	1
16	0	8	2
19	0	6	1
25	0	0	1

table 11

Theorem 3.4 *The systems BSTSs(51), (X, \mathcal{B}) , obtained by doubling plus one construction from a colorable BSTS(25), (X', \mathcal{B}') with $\mathcal{C}'_1 = (5, 10, 10)$ or $\mathcal{C}'_2 = (1, 4, 8, 12)$, have not extended colorings.*

Proof.

Table 11 shows all the solutions with respect to \mathcal{C}'_1 . From corollary 2.1 and proposition 2.2, none of them defines an extended coloring. There are not solutions of the system (1) with respect to \mathcal{C}'_2 , so the theorem follows. \square

BSTS($2v + 1$)	BSTS(55)	BSTS(79)	BSTS(87)
BSTS(v)	BSTS(27)	BSTS(39)	BSTS(43)
\mathcal{C}'_1	(1,4,10,12)	(1,8,12,18)	(1,10,12,20)
\mathcal{C}'_2	(2,5,6,14)	(2,6,13,18)	(4,4,17,18)
\mathcal{C}'_3	–	(4,6,9,20)	–
\mathcal{C}'_4	–	(1,2,8,8,20)	–
\mathcal{C}'_5	–	(1,4,4,10,20)	–

table 12

The next BSTSs obtained by doubling plus one construction are BSTSs(55), BSTSs(79) and BSTSs(87), and they are, respectively, constructed from BSTSs(27), BSTSs(39) and BSTSs(43). In table 12, by theorem 2.2, we can find colorings \mathcal{C}' of BSTSs(27), BSTSs(39) and BSTSs(43) that are extendible, i.e., colorings which define extended colorings. The colorings \mathcal{C}' are obtained using the conditions of theorem 1.3.

For systems BSTSs(63) obtained by doubling plus one construction from BSTSs(31), we know, from [7], that they have not extended colorings. Finally, the systems BSTSs(67) and BSTSs(75), obtained from the systems BSTSs(33) and BSTSs(37), respectively, do not admit extended colorings because none of the coloring \mathcal{C}' of the systems BSTSs(33) and BSTSs(37), gives solutions of system (1) (see [8]).

Theorem 3.5 *The systems BSTSs(55) and BSTSs(87) have extended 4-coloring, the systems BSTSs(79) have extended 4 and 5-colorings. The systems BSTSs(63), BSTSs(67), BSTSs(75) have not extended colorings. \square*

The systems BSTSs(91), obtained by doubling plus one construction from a BSTS(45), can have extended colorings with respect to the colorings $\mathcal{C}'_1 = (2, 8, 14, 21)$ and $\mathcal{C}'_2 = (4, 6, 13, 22)$. The colorings \mathcal{C}'_1 and \mathcal{C}'_2 are the unique colorings that verify the hypotheses of theorem 1.3 and in [2] it is proved their existence. In the tables 13 and 14 there are all the solutions with respect to \mathcal{C}'_1 and \mathcal{C}'_2 .

c_1	c_2	c_3	c_4
6	6	11	23
3	6	20	17
3	6	24	13
3	26	0	17
3	30	0	13
6	6	30	4
6	17	0	23
6	36	0	4
12	0	11	23
12	0	30	4
23	0	0	23
42	0	0	4

table 13

c_1	c_2	c_3	c_4
4	8	12	22
10	2	12	22
1	8	21	16
1	8	25	12
1	28	1	16
1	32	1	12
4	8	31	3
4	19	1	22
4	38	1	3
10	2	31	3
21	2	1	22
40	2	1	3

table 14

Theorem 3.6 *The systems $BSTSs(91)$, (X, \mathcal{B}) , obtained by doubling plus one construction from a $BSTS(45)$, (X', \mathcal{B}') , have extended 4-colorings of \mathcal{C}_2 .*

Proof.

In the table 13 there are all the solutions with respect to \mathcal{C}'_1 . From corollary 2.1 and propositions 2.2 and 2.3, we have that these solutions do not define extended colorings of \mathcal{C}'_1 .

In the table 14 there are all the solutions with respect to \mathcal{C}'_2 . The first solution defines an extended 4-coloring which is obtained by the factorization of table 22 with the following coloring classes $X_1 = \{1, 2, 3, 4, 46, 47, 48, 49\}$, $X_2 = \{5, 6, 7, 8, 9, 10, 50, 51, 52, 53, 54, 55, 56, 57\}$, $X_3 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69\}$, $X_4 = \{24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91\}$.

Suppose that the second solution of table 14 permits a factorization \mathcal{F} which defines an extended h -coloring of \mathcal{C}'_2 . We have that all the 24 non monochromatic pairs of vertices of X'' colored with the colors [2] and [3] have to be in 19 factors corresponding to the vertices $x_l \in X'$ colored with [2] and [3]. The two vertices of X'' colored with [2] define one monochromatic pair which is in only one factor of \mathcal{F} . Moreover, in all of the other factors, these two vertices have to be always in two different non monochromatic pairs. So 24 non monochromatic pairs of colors [2] and [3] are in 12 factors corresponding to the vertices $x_l \in X'$ of color [2] and [3]. Finally, if we consider at least six factors containing the non monochromatic pairs of colors [2] and [i] with $i \neq 3$, then this solution cannot define an extended coloring.

The remaining solutions of table 14, by corollary 2.1 and propositions 2.2 and 2.3, do not permit extended coloring of \mathcal{C}'_2 . \square

It remains to study the BSTSs(99) obtained by doubling plus one construction from a BSTS(49). A BSTS(49) can be colored with just five colorings: $\mathcal{C}'_1 = (12, 17, 20)$, $\mathcal{C}'_2 = (14, 14, 21)$, $\mathcal{C}'_3 = (1, 4, 4, 20, 20)$, $\mathcal{C}'_4 = (5, 6, 14, 24)$ and $\mathcal{C}'_5 = (2, 8, 18, 21)$. All these colorings verify the hypothesis of theorem 1.3 and their existence is proved in [2].

Theorem 3.7 *The BSTSs(99), (X, \mathcal{B}) , obtained by doubling plus one construction from a BSTSs(49), (X', \mathcal{B}') , have extended 4 and 5-colorings.*

Proof.

To simplify the proof, we will not list all the solutions of the system (1) with respect to the colorings \mathcal{C}'_1 , \mathcal{C}'_2 , \mathcal{C}'_3 , \mathcal{C}'_4 and \mathcal{C}'_5 , and we will also omit all the solutions which do not respect the necessary conditions defined in the previous section.

The colorings \mathcal{C}'_1 and \mathcal{C}'_2 do not admit solutions for the system (1), so BSTSs(99) have not extended 3-colorings.

There are 84 solutions of the system (1) with respect to \mathcal{C}'_3 , but from corollary 2.1 and propositions 2.2 and 2.3 only three among all the solutions are significative. They are: $(4, 1, 5, 19, 21)$, $(3, 2, 5, 20, 20)$ e $(0, 0, 12, 16, 22)$.

The solution $(4, 1, 5, 19, 21)$ cannot define an extended 5-coloring because this coloring would have more than one coloring class with odd cardinality, and this is not possible using theorem 8 of [10].

The solution $(3, 2, 5, 20, 20)$ does not admit an extended coloring because in a factorization of the vertices X'' it is not possible to find in the four factors corresponding to the vertices $x_l \in X'$ of color [2] all the three monochromatic pairs of color [1] with respect to the six non monochromatic pairs of colors [1] and [2].

The solution $(0, 0, 12, 16, 22)$ admits an extended 5-coloring defined by the factorization of vertices X'' in table 23 of the appendix. The coloring classes of this coloring are $X_1 = \{1\}$, $X_2 = \{2, 3, 4, 5\}$, $X_3 = \{6, 7, 8, 9, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61\}$, $X_4 = \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77\}$ e $X_5 = \{30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$.

There are 29 solutions of the system (1) with respect to \mathcal{C}'_4 , but only three of them are significative: $(5, 10, 10, 25)$, $(11, 4, 10, 25)$ and $(3, 11, 12, 24)$.

Suppose that the solution (5, 10, 10, 25) defines an extended coloring. The 50 non monochromatic pairs of vertices of X'' , colored with [1] and [2], have necessarily to be in the 11 factors corresponding to the vertices $x_l \in X'$ of colors [1] and [2] in the following way: 4 of these pairs have to be in 5 factors corresponding to the vertices $x_l \in X'$ of color [1] and 5 pairs in 6 factors tied to the vertices $x_l \in X'$ of color [2]. Necessarily the 300 monochromatic pairs of vertices of X'' of color [4] have to be in 25 factors corresponding to the vertices $x_l \in X'$ of colors [1], [2] and [3], exactly in 12 pairs for every factor. This disposition obliges to put only 43 monochromatic pairs of color [3] on the 45 factors corresponding to the vertices $x_l \in X'$ of colors [1] and [2], and this does not permit a correct extended coloring.

Suppose that the solution (11, 4, 10, 25) determines an extended coloring. The 44 non monochromatic pairs of the vertices of X'' , colored with [1] and [2], have necessarily to be in 11 factors corresponding to the vertices $x_l \in X'$ of colors [1] and [2] in following way: 4 of these pairs have to be in 5 factors corresponding with the vertices $x_l \in X'$ of color [1] and only 3 pairs have to be in 6 factors corresponding with the vertices $x_l \in X'$ of color [2]. The number of these pairs is 38, while all these pairs are 44, and this does not permit a correct extended coloring.

The solution (3, 11, 12, 24) gives an extended 4-coloring which is defined by the factorization of the vertices of X'' in the table 24 of the appendix. This extended coloring has the following coloring classes $X_1 = \{1, 2, 3, 4, 5, 50, 51, 52\}$, $X_2 = \{6, 7, 8, 9, 10, 11, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63\}$, $X_3 = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75\}$, $X_4 = \{26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$.

There are 27 solutions of the system (1) with respect to \mathcal{C}'_5 , but from corollary 2.1 and propositions 2.2 and 2.3, we have that only three of them are significative: (8, 2, 19, 21), (6, 4, 20, 20) and (10, 0, 20, 20).

The solution (8, 2, 19, 21) does not defines extended colorings because in X'' there are two vertices of color [2] and a odd number of vertices of colors [3] and [4].

The solution (6, 4, 20, 20) defines an extended 4-coloring by the factorization of X'' represented on table 25. This extended coloring has the following coloring classes $X_1 = \{1, 2, 50, 51, 52, 53, 54, 55\}$, $X_2 = \{3, 4, 5, 6, 7, 8, 9, 10, 56, 57, 58, 59\}$, $X_3 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79\}$, $X_4 = \{29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$.

The solution $(10, 0, 20, 20)$ defines an other extended 4-coloring represented by the factorization on the table 26 and having the following coloring classes $X_1 = \{1, 2, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59\}$, $X_2 = \{3, 4, 5, 6, 7, 8, 9, 10\}$, $X_3 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79\}$, $X_4 = \{29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$. \square

4 Extended colorings for BSTSs($2^h - 1$)

The problem connected to the determination of an extended colorings for the particular systems of type BSTS($2^h - 1$) was studied in [7], where the authors found necessary conditions for the existence of extended colorings. They also proved that extended colorings do not exist when $h < 10$. The problem to understand if there are extended colorings for $h \geq 10$ remains still open.

In the following theorems \bar{h} denotes the maximum value of h for which the system BSTS($2^{\bar{h}} - 1$) has not extended coloring.

Theorem 4.1 ([7]) *If \mathcal{C} be an extended \bar{h} -coloring of a BSTS($2^{\bar{h}+1} - 1$), then there exists at least one $c_i = 0$.* \square

Theorem 4.2 ([7]) *Let \mathcal{C} be an extended \bar{h} -coloring of a BSTS($2^{\bar{h}+1} - 1$), $c_i > 0$ and $c_l = 0$ for $l < i$ ($i = 1$ is possible), and let $c_j > 0$ ($j = i + 1$ is possible) and $c_k = 0$ for $i + 1 \leq k < j$. Then $c_{j+t} > 0$ for all $j + t > i + 1$.* \square

Notice that if there exists an extended \bar{h} -coloring of a BSTS($2^{\bar{h}+1} - 1$), then, by corollary 2.1, the connected solution of the system (1) necessarily has all the $c_i > 0$ even.

The following propositions permits to prove that the unique solutions of systems (1), which could define an extended colorings for BSTSs($2^h - 1$), are only the solutions defined in theorem 4.2, but with $c_1 = 0$.

Proposition 4.1 *There are not extended \bar{h} -colorings of a BSTS($2^{\bar{h}+1} - 1$) corresponding to a solution of type $(0, \dots, 0, c_i, c_{i+1}, \dots, c_{i+c}, 0, \dots, 0, c_j, \dots, c_{\bar{h}})$, where $c_p > 0$ for $i \leq p \leq i + c$ ($i = 1$ is possible) with $c \geq 1$ and $j \leq p \leq \bar{h}$.*

Proof.

Suppose that \mathcal{C} is an extended \bar{h} -coloring corresponding to a solution of the system (1) of type $(0, \dots, 0, c_i, c_{i+1}, \dots, c_{i+c}, 0, \dots, 0, c_j, \dots, c_{\bar{h}})$. It defines a factorization \mathcal{F} of the vertices in X'' . If $x' \in X''$ is a vertex colored with the color $[i]$ in \mathcal{C} , then it is in a monochromatic pair of color $[i]$ for every factor \mathcal{F} corresponding to the vertices $x_l \in X'$ of color $[q]$, where $1 \leq q \leq i-1$ and $i+c+1 \leq q \leq j-1$. From proposition 2.2, we have

$$2^{j-1} - 2^{i+c} + 2^{i-1} \leq c_i \leq 2^{i-1} + 2^i,$$

from which it follows

$$2^{j-1} \leq 2^{i+c} + 2^i = 2^c \cdot 2^i + 2^i = 2^i \cdot (2^c + 1).$$

If we define $r = j - (i + c)$, then we have

$$2^i \cdot 2^{c+r-1} \leq 2^i \cdot (2^c + 1).$$

Let us consider $r \geq 2$,

$$2^{c+1} \leq 2^{c+r-1} \leq 2^c + 1.$$

These last inequality is equivalent to $2^c \leq 1$, this is impossible because $c \geq 1$. Consider the case $r = 1$, we have a solution of type $(0, \dots, 0, c_i, c_{i+1}, \dots, c_{j-2}, 0, c_j, \dots, c_{\bar{h}})$ where $j \geq i+3$ because $r = 1$. We have the following inequalities

$$2^{j-1} - 2^{j-2} + 2^{i-1} \leq c_i \leq 2^{i-1} + 2^i,$$

so we obtain that $2^{j-2} \leq 2^i$, and it is true only if $j < i+3$, but this is not possible because $j \geq i+3$ and theorem is proved. \square

Proposition 4.2 *If \mathcal{C} is an extended \bar{h} -coloring of a BSTS($2^{\bar{h}+1} - 1$), then $c_1 = 0$.*

Proof.

From proposition 4.1 and theorem 4.2, we have that the unique possible extended colorings with $c_1 \neq 0$ can be connected to the solutions of the system (1) of type $(c_1, 0, \dots, 0, c_j, c_{j+1}, \dots, c_{\bar{h}})$. Suppose that this solution defines an extended coloring \mathcal{C} .

From proposition 2.2, it is $c_1 \leq 2^{j-1} + 1$, but c_1 has to be even, so $c_1 \leq 2^{j-1}$. Let $x' \in X''$ be a vertex which is colored with the color $[1]$ in \mathcal{C} . In the

factorization \mathcal{F} of X'' , defined by \mathcal{C} , x' is in one non monochromatic pair of colors $[1]$ and $[p]$, which is contained in the unique factor corresponding to the vertex $x_1 \in X'$ of color $[1]$, and in $2^{j-1} - 2$ monochromatic pairs colored with $[1]$ and all of them are contained in the factors corresponding to the vertices $x_l \in X'$ colored with the colors $[k]$, for $2 \leq k \leq j - 1$. Since c_1 has to be even, then $c_1 \geq 2^{j-1}$, i.e., $c_1 = 2^{j-1}$.

Let us consider one $c_{k'}$, with $k' \geq j$, from proposition 2.2 it has to be even, therefore we have that $c_{k'} \leq 2^{k'-1}$. Since $c_1 = 2^{j-1}$, then $x' \in X''$ of color $[1]$ can be in $2^{j-1} - 1$ monochromatic pairs of color $[1]$ where, in particular, $2^{j-1} - 2$ pairs are in the factors corresponding to the vertices $x_l \in X'$ colored with the colors $[k]$ with $2 \leq k \leq j - 1$ and only one is in a factor corresponding to a vertex $x_l \in X'$ colored with a color $[k']$ where $j \leq k' \leq \bar{h}$.

Therefore in all the factors corresponding to the vertices $x_l \in X'$ of color $[k']$, with $j \leq k' \leq \bar{h}$, x' is contained in at least $2^{k'-1} - 1$ different pairs colored with the colors $[1]$ and $[k']$, so we have $c_{k'} \geq 2^{k'-1} - 1$ and because $c_{k'}$ has to be even, then $c_{k'} = 2^{k'-1}$ for $j \leq k' \leq \bar{h}$. If we replace these values in the equations of system (1), we have:

$$\begin{cases} 2^{2j-2} + \sum_{k'=j}^{\bar{h}} 2^{2k'-2} + 2^j + \sum_{k'=j}^{\bar{h}} 2^{2k'-1} = 2^{2\bar{h}} \\ \sum_{i=1}^h c_i = 2^{\bar{h}}. \end{cases} \quad (2)$$

It is easy to check that the second equation of system (2) is true, while the first one is not verified because $2^j \neq 0$. Thus, the solution $(c_1, 0, \dots, 0, c_j, c_{j+1}, \dots, c_{\bar{h}})$ does not permit to define a extended coloring. \square

From the previous two propositions we have that the unique solutions of the system (1) defining extended colorings are only the solutions of theorem 4.2 with $c_1 \neq 0$. We can classify them in the following two typology:

1. $(0, \dots, 0, c_i, 0, \dots, 0, c_j, c_{j+1}, \dots, c_{\bar{h}})$;
2. $(0, \dots, 0, c_i, c_{i+1}, \dots, c_{\bar{h}})$.

The following theorem permits to determine an infinite class of BSTSS($2^h + 1$) with not extended colorings for any integer h .

Theorem 4.3 *For every $h \geq 1$ there exists a BSTS($2^h - 1$) which has not extended colorings.*

Proof. As it was proved in [7], there exists certainly a $\bar{h} > 1$ for which there is a $S' = \text{BSTS}(2^{\bar{h}} - 1)$, (X', \mathcal{B}') with not extended colorings. We are going to show that it is always possible to determine a $S = \text{BSTS}(2^{\bar{h}+1} - 1)$, (X, \mathcal{B}) , obtained by doubling plus one construction from S' , which does not admit extended colorings.

Let \mathcal{C}' be a \bar{h} -coloring of S' , we define a factorization \mathcal{F} of the vertex set X'' , with $X' \cap X'' = \emptyset$ and $|X''| = 2^{\bar{h}}$, which does not permit an extended coloring of the system S . If we consider $X''_h \subset X''$, with $|X''_h| = 2^{\bar{h}-1}$, then we can easily built $2^{\bar{h}-1}$ factors of \mathcal{F} such that in all of these factors there are not pairs containing two vertices of X''_h .

Let $\mathcal{F}^{(1)}$ be a factorization of the vertices of X''_h , it has $2^{\bar{h}-1} - 1$ factors, and every factor contains $2^{\bar{h}-2}$ pairs. Let $\mathcal{F}^{(2)}$ be the factorization of vertices $X'' - X''_h$ where $|X'' - X''_h| = 2^{\bar{h}-1}$.

If we label the factors of $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ and consider the k -th factor in $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ then the union of all the $2^{\bar{h}-2}$ pairs in these two factors defines a factor of \mathcal{F} . If we repeat this procedure for every k with $1 \leq k \leq 2^{\bar{h}-1} - 1$, then we obtain $2^{\bar{h}-1} - 1$ factors that together with the previous $2^{\bar{h}-1}$ factors define all the complete factorization \mathcal{F} .

Finally we obtain the system $S = \text{BSTS}(2^{\bar{h}+1} - 1)$, by doubling plus one construction from S' , in the following way: the $2^{\bar{h}-1} - 1$ factors, obtained from the union of the pairs in the factors of $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$, are connected with the vertices $x_i \in X'$ colored with the colors $[j]$ with $1 \leq j \leq \bar{h} - 1$ in \mathcal{C}' , while all the other factors of \mathcal{F} are connected with the vertices $x_i \in X'$ of color $[\bar{h}]$ in \mathcal{C}' .

In this system, because S' has not extended coloring we have that the pairs in $\mathcal{F}^{(2)}$ cannot be colored with any color $[i]$, for $1 \leq i \leq \bar{h} - 1$, of \mathcal{C}' and the system S has not extended coloring. It is clear that recursively, for every integer h , it is always possible to obtain a $\text{BSTS}(2^h - 1)$ which has not extended colorings and the theorem follows. \square

5 Concluding remarks

In this paper, we determine particular strict colorings, called extended colorings, of particular $\text{BSTS}(2v+1)$, obtained by doubling plus one construction from a h -colorable $\text{BSTS}(v)$. If \mathcal{C}' is an edge coloring of $\text{BSTS}(v)$, then the $\text{BSTS}(2v+1)$ has an extended coloring \mathcal{C} of \mathcal{C}' if the subsystem $\text{BSTS}(v)$ is colored with \mathcal{C}' .

We prove that the unique systems $BSTSs(v)$, for $v < 103$, with extended colorings are the systems whose order are in the following set $V = \{27, 39, 43, 55, 79, 87, 91, 99\}$. The table 15 resumes all the results obtained in section 3, i.e., it contains all the $BSTSs$, of order less than 103, which have extended h -coloring for all the possible h .

If a $BSTS(v)$ can be colored with an extended coloring, then we have that the upper and lower chromatic number have distinct values, i.e., $\chi \neq \bar{\chi}$. The existence of these particular colorings gives a useful tool to study all the possible different kind of feseable sets for a single colorable $BSTS(v)$, where a feseable set of a $BSTS(v)$ is the set $\Omega(v) = \{k : \text{for which there exists a } k\text{-coloring}\}$ (see [8]). In particular, all the $BSTSs(v)$ with $v \in V$ and with extended coloring permit to define infinite classes of $BSTSs$ which have two strict colorings with a different number of colors. Theorem 5.1 gives this property and it can be proved easily repeating a sequence of doubling plus one constructions.

$BSTS(v)$	3-colorings	4-colorings	5-colorings
$BSTS(27)$	yes	no	no
$BSTS(39)$	yes	yes	no
$BSTS(43)$	yes	no	no
$BSTS(55)$	no	yes	no
$BSTS(79)$	no	yes	yes
$BSTS(87)$	no	yes	no
$BSTS(91)$	no	yes	no
$BSTS(99)$	no	yes	yes

table 15

Theorem 5.1 *If a $BSTS(v)$ with $v \in \{27, 39, 43, 55, 79, 87, 91, 99\}$ has an extended h -colorings, then there exists an infinite class of $BSTSs(2^k(v+1) - 1)$ with at least two $(h+k-1)$ e $(h+k)$ -colorings, and in particular we have that $\chi \neq \bar{\chi}$. \square*

For the systems of type $BSTSs(2^h - 1)$ we give new necessary and sufficient conditions for finding extending colorings. In particular, we obtain that the solutions of the system (1) which define extended colorings can only be of two types: $(0, \dots, 0, c_i, 0, \dots, 0, c_j, c_{j+1}, \dots, c_{\bar{h}})$ and $(0, \dots, 0, c_i, c_{i+1}, \dots, c_{\bar{h}})$. It is important to underline that extended coloring have been finding for these particular systems and the following conjecture is still true [6].

Conjecture 5.1 ([6]) *If a $BSTS(2^h - 1)$ is obtained by a sequence of doubles plus one constructions from $BSTS(3)$, then it has not extended colorings for every $h \geq 1$.*

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Extended colorings for $BSTS(2v + 1)$

M. Gionfriddo*

E. Guardo*

L. Milazzo*

28 June 2011

In memory of Lucia Gionfriddo

Abstract

A $BSTS(2v + 1)$ containing a colorable subsystem $BSTS(v)$ with h -coloring \mathcal{C}' has an extended h -coloring of \mathcal{C}' if it is also h -colorable with a coloring \mathcal{C} in which the subsystem $BSTS(v)$ is colored with \mathcal{C}' . In this paper we give both necessary conditions and sufficient conditions for the existence of an extended coloring. The existence of these colorings is studied either for systems of type $BSTSs(2v + 1)$ containing subsystems $BSTSs(v)$ with $2v + 1 < 103$ or systems of type $BSTSs(2^h + 1)$.

1 Introduction

A triple system $STS(v)$ is defined by a pair (X, \mathcal{B}) , where X is a finite set of vertices with $|X| = v$, and \mathcal{B} is a family of subsets of X , called *blocks*, such that each block contains only three vertices, and any two distinct vertices of X appear together in precisely only one block in \mathcal{B} . It is well known that it must be $v \equiv 1 \text{ or } 3 \pmod{6}$ (see [6]).

A proper k -coloring of (X, \mathcal{B}) is a mapping ϕ from X into a finite set C of k colors, $\{1, 2, \dots, k\}$. In 1993, in [16, 17], V. Voloshin introduced a new coloring for a mixed hypergraph. A mixed hypergraph is a triple $H = (X; \mathcal{C}; \mathcal{D})$, where X is a finite set of vertices, while \mathcal{C} and \mathcal{D} are two families of subsets of X . The elements of \mathcal{C} and \mathcal{D} are called \mathcal{C} -edges and \mathcal{D} -edges respectively. If $\mathcal{C} = \emptyset$, then \mathcal{H} is called a \mathcal{D} -hypergraph, while if $\mathcal{D} = \emptyset$ then \mathcal{H} is called a \mathcal{C} -hypergraph. A strict k -coloring of \mathcal{H} is a vertex coloring

* Department of Mathematics and Informatics, University of Catania, Viale A. Doria, 6 95125 - Catania, Italy. E-mail: gionfriddo@dmf.unict.it, guardo@dmf.unict.it, milazzo@dmf.unict.it.

where any \mathcal{C} -edge has at least two vertices of the same color and any \mathcal{D} -edge has at least two vertices colored differently, and exactly k colors are used in it. If it is not necessary to know the number of used colors then a strict k -coloring will be called strict coloring.

The minimum k for which there exists a strict k -coloring is called the lower chromatic number of \mathcal{H} and is denoted by χ . The strict coloring allow us to characterize the maximum number k for which there exists a strict k -coloring. It is called the upper chromatic number of \mathcal{H} and is denoted by $\bar{\chi}$. If there exists no strict coloring of \mathcal{H} , then \mathcal{H} is said to be uncolorable.

In a \mathcal{D} -hypergraph, the lower chromatic number coincides with the (weak) chromatic number (see [2, 4]) and the upper chromatic number trivially equals v . In a \mathcal{C} -hypergraph, the lower chromatic number trivially equals 1, but the upper chromatic number represents a value that is hard to determine. Mixed hypergraphs with $\mathcal{D} = \mathcal{C}$ are called bi-hypergraphs, and the subsets of X in consideration are called bi-edges. In any proper coloring of bi-hypergraphs, each bi-edge is neither monochromatic (because it is a \mathcal{D} -edge) nor polychromatic (because it is a \mathcal{C} -edge).

In [10], the authors determine hypergraphs that are uncolorable; this problem of uncolorability did not originally exist in the theory of hypergraph coloring (see [2]), as it arises only because of the interaction between \mathcal{D} -edges and \mathcal{C} -edges when a proper coloring is sought and it is called *strong interaction*.

For k -colorable hypergraph \mathcal{H} , let r_k be the number of partitions of the vertex set into k nonempty parts (color classes) such that the coloring satisfies the coloring of each \mathcal{C} -edge and each \mathcal{D} -edge. In this case r_k coincides with the number of strict k -colorings if we do not count permutations of the colors.

The integer vector

$$R(\mathcal{H}) = (r_1, \dots, r_k),$$

is called *chromatic spectrum*. The chromatic spectrum, introduced in [16], can be broken (it may have gaps), i.e., it may happen that $r_i = 0$ for some $\chi < i < \bar{\chi}$. The existence of a gap is due to *weak interactions* between \mathcal{C} -edges and \mathcal{D} -edges. The problem of identifying an $STS(v)$ with broken spectrum has not been determined and the only known designs with gaps in the chromatic spectrum are the BP_3 -designs which were found in [5] and the BP_4 -designs found in [1].

In [13], the authors determine two types of coloring of $STS(v)$: 1) $CSTSs(v)$ colorings where each block is a \mathcal{C} -edge and the vertices of each single block

can be colored with one or two colors; 2) $BSTSs(v)$ colorings where each block is both a \mathcal{C} -edge and a \mathcal{B} -edge, and the vertices of a block can be colored only with two colors.

The following two theorems are very important in $CSTSs(v)$ and $BSTSs(v)$ coloring's theory, since they determine the best upper bound of its upper chromatic number, a lower bound for the cardinality of the color classes, and necessary conditions for $BSTS$ colorings.

Theorem 1.1 ([13]) *If S is a $CSTS(v)$ or a $BSTS(v)$ of order $v \leq 2^h - 1$ ($h \in \mathbb{N}$), then $\bar{\chi}(S) \leq h$. \square*

Theorem 1.2 ([15]) *If \mathcal{C} is a strict coloring of a $BSTS$ or $CSTS$ using h colors, then $n_i \geq 2^{i-1}$ for all $1 \leq i \leq h$. \square*

Theorem 1.3 ([11]) *If $S = BSTS(v)$ is colorable with a k -coloring, then the following conditions are true*

$$s_i(s_i - 1) \leq 3 \sum_{j=1}^i n_j(n_j - 1); \quad (i)$$

$$s_k(s_k - 1) = 3 \sum_{j=1}^k n_j(n_j - 1), \quad (k)$$

for every and $1 < i \leq k$. \square

The systems $STSs(2v+1)$ that are obtained by a *doubling plus one construction* are fundamental in the $CSTSs(v)$ and $BSTSs(v)$ coloring's theory. A system $STS(2v+1)$, (X, \mathcal{B}) is obtained from a $STS(v)$, (X', \mathcal{B}') by a doubling plus one construction, if each element of X' can be represented by a symbol x_i for $1 \leq i \leq v$ and if there exists an other set of vertices X'' such that $|X''| = v+1$ and $X' \cap X'' = \emptyset$, and $\mathcal{F} = (F_1, F_2, \dots, F_v)$ is a factorization of the vertices of X'' . The set \mathcal{B} contains all the blocks of \mathcal{B}' and also the blocks of type $\{x_i, y_l, y_m\}$ where $x_i \in X'$ and $(y_l, y_m) \in F_i$. If we consider the set of vertices $X = X' \cup X''$, then the pair (X, \mathcal{B}) defines a $STS(2v+1)$ since $|\mathcal{B}| = \frac{2v(2v+1)}{6}$ and any pair of vertices of X is in one and only one block of \mathcal{B} . The following theorem shows that the unique $CSTSs(v)$ and $BSTSs(v)$ with $\bar{\chi} = h$ and $v \leq 2^h - 1$ are obtained from $STS(3)$ by repeated applications of doubling plus one constructions.

Theorem 1.4 ([13]) *If S is a $CSTS(v)$ or a $BSTS(v)$ with $v \leq 2^h - 1$ and $\bar{\chi} = h$ then:*

1. $v = 2^h - 1$;
2. *in any strict coloring of S with h colors, the color classes have cardinalities*

$$2^0, 2^1, 2^2, \dots, 2^{h-1},$$

and all of them are stable sets;

3. *S is obtained from the $STS(3)$ by repeated applications of doubling plus one constructions.* □

In this paper we study particular colorings of $BSTSs(v)$ obtained by doubling plus one construction, where each block can be colored with only two colors.

Let $S = BSTS(2v+1)$, (X, \mathcal{B}) be obtained by a doubling plus one construction from $S' = BSTS(v)$, (X', \mathcal{B}') which is h -colorable with the coloring \mathcal{C}' . We say that the system $S = BSTS(2v+1)$ has an *h -extended coloring of \mathcal{C}'* if there exists a h -coloring \mathcal{C} of S such that the subsystem S' is colored with \mathcal{C}' . This is equivalent to say that the h -coloring \mathcal{C} of S depends only on the coloring of the vertices of $X'' = X - X'$ by the colors of \mathcal{C}' . In [8] the authors proved that extended h -colorings don't exist if $v = 2^h - 1$ and $h < 10$, and it leaves as an open problem the case where $h \geq 10$; in particular, using theorems 1.1 and 1.4, this last result allows us to get information on the lower and upper chromatic numbers.

Theorem 1.5 ([8]) *If a $BSTS(2^h - 1)$ or a $CSTS(2^h - 1)$ is obtained by a sequence of doubling constructions plus one starting from $STS(3)$, then $\bar{\chi} = \chi = h$ for all $h < 10$.* □

Our paper is structured as follows. In Section 2 we give necessary conditions for the existence of extended h -colorings for generic systems $BSTSs(2v+1)$. We also give a sufficient condition to determine particular extended colorings. Section 3 is devoted to determine which systems $BSTSs(2v+1)$, with $2v+1 < 103$, obtained from $BSTS(v)$ by doubling plus one construction, have extended colorings. In Section 4 we give necessary conditions for the

existence of extended colorings of particular systems $BSTS(2^h - 1)$. For each of them, we classify the cardinalities of the color classes of possible extended colorings and for every $h \geq 1$, we also determine an infinite class of $BSTS(2^h - 1)$ that cannot be colored by extended colorings.

2 Extended colorings

Let $S' = BSTS(v)$, (X', \mathcal{B}') be a h -colorable system with a coloring $\mathcal{C}' = \{n'_1, n'_2, \dots, n'_h\}$ and let $S = BSTS(2v + 1)$, (X, \mathcal{B}) be a system obtained by doubling plus one construction plus one from S' . The system S is certainly $(h + 1)$ -colorable with a coloring $\mathcal{C}'' = \{n''_1, n''_2, \dots, n''_h, n''_{h+1}\}$ where $n''_{h+1} = v + 1$. This coloring is obtained coloring the subsystem S' with \mathcal{C}' and the vertices in $X'' = X - X'$, where $|X''| = v + 1$, are colored using a new color, denoted by $[h + 1]$, different from the h colors in \mathcal{C}' . We are considering the problem to color the system S with an extended h -coloring $\mathcal{C} = \{n_1, n_2, \dots, n_h\}$ where the vertices of the subsystem S' are colored with \mathcal{C}' . This extended coloring depends only on the colors assigned to the vertices in X'' , and on the fact that they are colored with just the colors used in \mathcal{C}' .

The values $c_i = n_i - n'_i$, with $1 \leq i \leq h$, are the numbers of vertices in X'' which are colored with the color $[i] \in \mathcal{C}'$. It is possible that $c_j = 0$, for some $1 \leq j \leq h$, and it is evident that $\sum_{i=1}^h c_i = v + 1$. A necessary condition for the existence of an extended h -coloring of \mathcal{C}' for S is given by following theorem.

Theorem 2.1 *Let $S = BSTS(2v + 1)$, (X, \mathcal{B}) be a system obtained by doubling plus one construction from $S' = BSTS(v)$, (X', \mathcal{B}') h -colorable with the coloring $\mathcal{C}' = \{n'_1, n'_2, \dots, n'_h\}$. If $\mathcal{C} = \{n_1, n_2, \dots, n_h\}$ is an extended h -coloring of \mathcal{C}' for S where $n_i = n'_i + c_i$, then the two following equations are both satisfied:*

$$\begin{cases} \sum_{i=1}^h c_i^2 + 2 \sum_{i=1}^h n'_i c_i = (v + 1)^2 \\ \sum_{i=1}^h c_i = v + 1. \end{cases} \quad (1)$$

Proof.

Since we know that $\sum_{i=1}^h c_i = v + 1$, we need to prove only the first equality of (1). Let \mathcal{F} be the factorization of $X'' = X - X'$ defined in the doubling plus one construction from which we get S , and let $F_l \in \mathcal{F}$ be one of its factor corresponding to the vertex $x_l \in X'$ colored with a color $[i]$.

In the factorization \mathcal{F} , the numbers of different monochromatic and non monochromatic pairs are $\sum_{i=1}^h \binom{c_i}{2}$ and $\sum_{i=1}^h n_i c_i$, respectively. A generic monochromatic pair colored with the color $[j]$ has to be in a factor corresponding to the vertices $x_l \in X'$ colored with $[k]$, where $[j] \neq [k]$. Since in a doubling plus one construction any two vertices of X'' are in only one block which contains a vertex of X' , then a generic non monochromatic pair colored with $[j]$ and $[k]$ has to be in a factor corresponding to a vertex $x_l \in X'$ colored either with the color $[j]$ or $[k]$. If we consider $c_j \neq 0$, for some j , then in all of the factors corresponding to the vertices $x_l \in X'$ of color $[j]$ there are exactly c_j non monochromatic pairs.

In \mathcal{F} , the number of all pairs is $\frac{v(v+1)}{2}$ and, therefore, we have that

$$\sum_{i=1}^h \binom{c_i}{2} + \sum_{i=1}^h n'_i c_i = \frac{v(v+1)}{2}.$$

It easy to check, by simple calculation, that we obtain the first equality of (1), and hence, the theorem follows. \square

The solutions of the system (1) given in theorem 2.1 are called *solutions with respect to \mathcal{C}'* . It is important to note that this theorem gives only necessary conditions for the existence of an extended coloring; indeed, in [8] the authors determine solutions of the system (1) for $BSTS_s(2^h - 1)$, with $6 \leq h < 10$, but these solutions do not define any extended coloring.

The following corollary gives other conditions useful to find solutions for the system (1) which could define extended colorings.

Corollary 2.1 *Let $S = BSTS(2v+1)$ be a system obtained by doubling plus one construction from a h -colorable $S' = BSTS(v)$ with the coloring $\mathcal{C}' = \{n'_1, n'_2, \dots, n'_h\}$, and let $\{c_1, c_2, \dots, c_h\}$ be a solution with respect to \mathcal{C}' .*

1. *If there exists a $c_j = 0$ then all the $c_i > 0$ are even.*
2. *If there exists a $c_j > \frac{v+1}{2}$ then it does not exist an extended coloring of \mathcal{C}' .*

Proof.

1. If there exists a $c_i = 0$, then in all of the factors corresponding to the vertices $x_l \in X'$ colored with $[i]$ there are all the monochromatic pairs, so every $c_k > 0$ has to be even.
2. if there exists a $c_j > \frac{v+1}{2}$, then in all of the factors corresponding to the vertices $x_l \in X'$ colored with $[j]$ there are monochromatic pairs of color $[j]$,

and this is not possible. \square

The first $BSTSs(v)$ which can be obtained by doubling plus one construction are $BSTSs(7)$ and $BSTSs(15)$. But, as it was proved in [8], these systems have not solutions with respect to $\mathcal{C}' = \{1, 2\}$ and to $\mathcal{C}'' = \{1, 2, 2^2\}$, so they have not extended colorings. The next systems are $BSTSs(19)$ obtained by doubling plus one construction from $BSTS(9)$. This latter system can be colored with an unique 3-coloring $\mathcal{C}' = \{1, 4, 4\}$ (see [14]). All the solutions with respect to \mathcal{C}' are in the table 1.

c_1	c_2	c_3
3	2	5
3	5	2
5	0	5
5	5	0
8	0	2
8	2	0

table 1

From corollary 2.1, all the solutions of table 1, with the exception of the first and the second ones, do not permit to obtain extended colorings. In the first solution, since $c_1 = 3$ and $c_3 = 5$, it is necessary that in the four factors corresponding to the vertices $x_l \in X'$ colored with [2], there are exactly two non monochromatic pairs colored with [1] and [2] and with [2] and [3], one monochromatic pair of color [1] and two monochromatic pairs of color [3]. Because $c_1 = 3$ we have that these dispositions are not permitted, so this solution does not permit to define an extended coloring. Analogously, it is possible to prove that also the second solution of table 1 does not permit an extended coloring, therefore there are not extended colorings for $BSTSs(19)$. In [11] it was proved that there are $BSTSs(19)$ uniquely 3-colorable, uniquely 4-colorable and 3 and 4-colorable, so we can consider the following proposition.

Proposition 2.1 *All the $BSTSs(19)$, which are 3 and 4-colorable or uniquely 3-colorable, do not contain a subsystem $BSTS(9)$.* \square

The following theorem gives a sufficient condition for the existence of extended h -colorings of $BSTS(2v+1)$ when there is a h -colorings \mathcal{C}' of system $BSTS(v)$, and it permits to us to find extended colorings without resolving system (1).

Theorem 2.2 *Let $S' = BSTS(v)$, (X', \mathcal{B}') be a system h -colorable with $\mathcal{C}' = (n'_1, n'_2, \dots, n'_h)$. If there exist p integers n'_{k_i} , with $1 \leq i \leq p$ and $h > p$, where $n'_{k_1} + n'_{k_2} = (v+1)/2^{p-1}$ is an even integer, and $n'_{k_i} = (v+1)/2^{p-i+1}$, for $3 \leq i \leq p$, are all even, then $S = BSTS(2v+1)$, (X, \mathcal{B}) , obtained by doubling construction plus one from S' , has an extended h -coloring of \mathcal{C}' .*

Proof.

Set $c_{k_1} = (v+1)/2^{p-1}$, $c_{k_2} = (v+1)/2^{p-1}$, $c_{k_i} = (v+1)/2^{p-i+1}$ for $3 \leq i \leq p$, and $c_j = 0$ for all $j \neq k_i$ with $1 \leq i \leq p$. It is necessary that these assignments give correct numbers of monochromatic and non monochromatic pairs in a factorization \mathcal{F} of X'' which defines a coloring of $BSTS(2v+1)$, i.e., it is necessary to check if (c_1, c_2, \dots, c_h) is a solution with respect to \mathcal{C}' of the system

$$\begin{cases} \sum_{i=1}^p c_{k_i}^2 + 2 \sum_{i=1}^p n'_{k_i} c_{k_i} = (v+1)^2 \\ \sum_{i=1}^p c_{k_i} = (v+1). \end{cases}$$

Replacing the values of c_{k_i} and n'_{k_i} , for $1 \leq i \leq p$, the first equality becomes

$$\sum_{i=1}^{2p-2} \frac{(v+1)^2}{2^i} + \frac{(v+1)^2}{2^{2p-2}} = (v+1)^2,$$

and it is trivially true.

Set $S = \sum_{i=2}^p (v+1)/2^{p-i+1} + (v+1)/2^{p-1}$. It is simple to verify that the difference $S = 2S - S$ is equal to $v+1$, so also the second equality is true. In this second part of the proof, we are going to specify the modalities of the distribution of the vertices in X'' and of the colors $[k_i]$, with $1 \leq i \leq h$, in a factorization \mathcal{F} in such a way the rules of an extended coloring of \mathcal{C}' are respected.

Let us color exactly $(v+1)/2$ vertices with the color $[k_p]$ and the others $(v+1)/2$ with the colors $[k_i]$ with $1 \leq i \leq p-1$. Easily, it is possible to build $(v+1)/2$ factors with non monochromatic pairs using for each one of them colors $[k_p]$ and $[k_i]$, with $1 \leq i \leq p-1$. These factors are connected with the $(v+1)/2$ vertices $x_l \in X'$ colored with $[k_p]$. The set of the $(v+1)/2$ vertices in X'' colored with $[k_p]$ defines a factorization $\mathcal{F}^{(1)}$ of $(v-1)/2$ factors all containing $(v+1)/2^2$ monochromatic pairs of color $[k_p]$. The factors in $\mathcal{F}^{(1)}$ are placed on the bottom of the $(v-1)/2$ factors in \mathcal{F} corresponding to the vertices $x_l \in X'$ colored with all the colors distinct from $[k_p]$. Now, let us consider $(v+1)/2^2$ vertices of X'' colored with the color $[k_{p-1}]$; all the non monochromatic pairs that use the colors

$[k_{p-1}]$ and $[k_i]$, with $1 \leq i \leq p-2$, define $(v+1)/2^2$ factors all containing $(v+1)/2^2$ pairs. These last factors cover completely $(v+1)/2^2$ factors of \mathcal{F} which contain $(v+1)/2^2$ monochromatic pairs of color $[k_p]$ of $\mathcal{F}^{(1)}$, and they are connected with the vertices $x_l \in X'$ colored with $[k_{p-1}]$. The $(v+1)/2^2$ vertices of X'' colored with $[k_{p-1}]$ define a factorization $\mathcal{F}^{(2)}$ of $(v+1)/2^2 - 1$ factors. The pairs of these factors are added to the other incomplete factors of \mathcal{F} which contain $(v+1)/2^2$ monochromatic pairs colored with $[k_p]$ and contained in the factors of $\mathcal{F}^{(1)}$. Therefore in these last factors there are $(v+1)/2^2$ pairs of color $[k_p]$ and $(v+1)/2^4$ pairs of color $[k_{p-1}]$. We repeat this procedure until that the $(v+1)/2^{p-2}$ vertices in X'' colored with $[k_3]$ define $(v+1)/2^{p-2}$ factors of non monochromatic pair colored with the colors $[k_3]$ and $[k_i]$, with $i = 1$ and 2 . They completely cover $(v+1)/2^{p-2}$ factors of \mathcal{F} corresponding to the vertices $x_l \in X'$ colored with $[k_3]$. Also the vertices of X'' colored with $[k_3]$ define a factorization $\mathcal{F}^{(p-2)}$ of $(v+1)/2^{p-2} - 1$ factors of $(v+1)/2^{p-1}$ monochromatic pairs colored with $[k_3]$. The factors in $\mathcal{F}^{(p-2)}$ are posed on the remanning $(v+1)/2^{p-2} - 1$ factors of \mathcal{F} which are not complete and containing monochromatic pairs of colors $[k_i]$ for $3 \leq i \leq p$. Finally the $(v+1)/2^{p-1}$ vertices of color $[k_1]$ and the $(v+1)/2^{p-1}$ vertices of colors $[k_2]$ define $(v+1)/2^{p-1}$ containing non monochromatic pairs and colored with $[k_1]$ and $[k_2]$. These factors completely cover all the $(v+1)/2^{p-1}$ factors of \mathcal{F} corresponding with the vertices $x_l \in X'$ colored with $[k_1]$ and $[k_2]$. The vertices of X'' colored with $[k_1]$ and $[k_2]$ define respectively two factorizations $\mathcal{F}^{(p-1)}$ and $\mathcal{F}^{(p)}$ of $(v+1)/2^{p-1} - 1$ factors all containing $(v+1)/2^p$ monochromatic pairs of colors $[k_1]$ and $[k_2]$. These factors cover completely all the remanning $(v+1)/2^{p-1} - 1$ factors of \mathcal{F} corresponding to the vertices $x_l \in X'$ colored with $[j] \neq [k_i]$ with $1 \leq i \leq p$. We obtain a factorization \mathcal{F} of $\sum_{i=1}^{p-1} (v+1)/2^i + (v+1)/2^{p-1} - 1 = v$ factors which gives a correct coloring for the BSTS($2v+1$) obtained by doubling construction plus one and the theorem follows. \square

In the previous theorem it is possible to obtain the factorization \mathcal{F} since all quantities $(v+1)/2^i$ with $1 \leq i \leq p-1$ are even, therefore it is always possible to construct the factorizations $\mathcal{F}^{(i)}$, for $1 \leq i \leq p-1$. Notice that in this theorem the solution with respect to \mathcal{C}' has at least a $c_l = 0$ with $1 \leq l \leq h$, since in \mathcal{F} there are factors with only monochromatic pairs. The following corollary fixes the particular case when $p = 2$.

Corollary 2.2 *Let $S' = \text{BSTS}(v)$ be an h -colorable system with the coloring $\mathcal{C}' = (n_1, n_2, \dots, n_h)$, if there are two n_i and n_j such that $n_i + n_j = \frac{(v+1)}{2}$ is even, then the BSTS($2v+1$) obtained by doubling construction plus one*

from S' has an extended coloring with respect to \mathcal{C}' .

Proof.

We obtain the proof using the same technique of theorem 2.2 assigning the color $[i]$ to $(v+1)/2$ vertices of X'' and the color $[j]$ to the remaining vertices. \square

The following corollary allows us to determine infinite classes of BSTSs having extended colorings.

Corollary 2.3 *Let $S' = \text{BSTS}(v)$, (X', \mathcal{B}') be a system h -colorable with the coloring $\mathcal{C}' = (n'_1, n'_2, \dots, n'_h)$. If there exist p integers n'_{k_i} , with $1 \leq i \leq p$, such that $n'_{k_1} + n'_{k_2} = (v+1)/2^{p-1}$ is even and $n'_{k_i} = (v+1)/2^{p-i+1}$, for $3 \leq i \leq p$, are all even, then all the BSTSs $(2^k \cdot (v+1) - 1)$, with k positive integer and obtained from a sequence of doubling constructions plus one from S' , have an extended $(h+k-1)$ -coloring.*

Proof.

In general every $\text{BSTS}(2^k \cdot (v+1) - 1)$ has an extended $(h+k-1)$ -coloring with respect to the coloring $\mathcal{C}'' = (n''_1, n''_2, \dots, n''_h, n''_{h+1}, \dots, n''_{h+k-1})$ of the system $\text{BSTS}(2^{k-1} \cdot (v+1) - 1)$ where $n''_i = n'_i$, with $1 \leq i \leq h$, and $n''_j = 2^{j-h-1} \cdot (v+1)$ with $h+1 \leq j \leq h+k-1$, therefore by theorem 2.2 the corollary is true. \square

The first following proposition gives a necessary condition for the existence of an extended colorings connected with a solutions of the system (1) with respect to the coloring \mathcal{C}' , while the second one permits to characterize solutions of system (1) which do not give extended colorings.

Proposition 2.2 *Let \mathcal{C} be an extended coloring of $\mathcal{C}' = (n'_1, n'_2, \dots, n'_h)$ for the system $S = \text{BSTS}(2v+1)$, (X, \mathcal{B}) , obtained by a doubling plus one construction from a $S' = \text{BSTS}(v)$, (X', \mathcal{B}') . If in a solution (c_1, c_2, \dots, c_h) with respect to \mathcal{C}' there are two $c_i > 0$ and $c_j > 0$, with $1 \leq i, j \leq h$ and $i \neq j$, then we have that $c_i \leq n'_i + n'_j$ and $c_j \leq n'_i + n'_j$.*

Proof.

Let us consider $x' \in X''$ colored with $[i]$, it is in c_j non monochromatic pairs colored with $[i]$ and $[j]$ all contained in different factors. These factors are corresponding to the vertices $x_l \in X'$ colored only with either $[i]$ or $[j]$ and they are at most $n'_i + n'_j$, so $c_j \leq n'_i + n'_j$. Analogously, we obtain that

$$c_i \leq n'_i + n'_j. \quad \square$$

The previous proposition has the best utility when in \mathcal{C}' there is a $n'_i = 1$ and $c_i > 0$, in fact in this case $c_j \leq +n'_j + 1$ for every $c_j > 0$ with $1 \leq j \leq h$ and $i \neq j$. In the general case, if we want the best evaluation of a solution of system (1) with respect to the conditions of proposition 2.2, then we have to find in \mathcal{C}' the value of n'_i such that $c_i > 0$, $n'_i \leq n'_j$ for every $1 \leq j \leq h$, with $c_j > 0$ and $i \neq j$. This particular choice of n'_i permits to optimize the inequalities $c_j \leq +n'_j + n'_i$ with $i \neq j$.

Proposition 2.3 *Let $S = BSTS(2v+1)$, (X, \mathcal{B}) be a system obtained by doubling plus one construction from the system $S' = BSTS(v)$, (X', \mathcal{B}') , colorable with the coloring $\mathcal{C}' = (n'_1, n'_2, \dots, n'_h)$. If (c_1, c_2, \dots, c_h) is a solution of system (1) with respect to \mathcal{C}' , with $c_l > 0$ for $1 \leq l \leq h$, and $c_i = (v+1)/2$, and with a $c_j > 0$ such that $(\sum_k n'_k) \cdot \lfloor c_j/2 \rfloor < c_j(c_j-1)/2$, with $k \neq i$ and j , then the solution (c_1, c_2, \dots, c_h) does not determine an extended h -coloring \mathcal{C}' .*

Proof.

In X'' , the solution (c_1, c_2, \dots, c_h) defines $c_j(c_j-1)/2$ monochromatic pairs of color $[j]$ which are not in the factors corresponding to the vertices $x_l \in X'$ colored with $[i]$ and $[j]$. These pairs have to be in $\sum_k n'_k$ factors, with $k \neq i, j$ and corresponding to the vertices $x_l \in X'$ of color $[k]$. In these factors there are at most $\lfloor c_j/2 \rfloor$ monochromatic pairs of color $[j]$, therefore if $(\sum_k n'_k) \cdot \lfloor c_j/2 \rfloor < c_j(c_j-1)/2$, then it is not possible to obtain an extended coloring. \square

3 Extended colorings for small BSTSs

In the previous section it was proved that there are not extended colorings for $BSTS(v)$ with $v \leq 19$. In this section we study the first systems with extended colorings.

Consider the system $BSTS(27)$ obtained by doubling plus one construction from $BSTS(13)$ which can be uniquely 3-colorable with $\mathcal{C}' = (2, 5, 6)$ (see [12]). All the solutions of the system (1) with respect to \mathcal{C}' are in the table 2.

c_1	c_2	c_3
4	4	6
7	1	6
4	7	3
7	7	0
10	1	3
10	4	0

table 2

Theorem 3.1 *There exists a BSTS(27), (X, \mathcal{B}) , obtained by doubling plus one construction from one BSTS(13), (X', \mathcal{B}') , which has an extended 3-coloring of $\mathcal{C}' = (2, 5, 6)$. For this system we have that $\chi = 3$ e $\bar{\chi} = 4$.*

Proof.

Table 2 shows all the six solutions of the system (1) with respect to the coloring \mathcal{C}' . From corollary 2.1, the last three solutions of this table do not define extended colorings of \mathcal{C}' .

Consider the solution $(4, 7, 3)$, there are three monochromatic pairs of color [3]. We can put two of them in two factors of \mathcal{F} corresponding to the two vertices $x_l \in X'$ of color [1], but the third one can be neither in the five factors corresponding to the vertices $x_l \in X'$ of color [2] nor in the six factors corresponding to the vertices $x_l \in X'$ of color [3], so this solution cannot define an extended coloring of \mathcal{C}' .

The solutions $(4, 4, 6)$ and $(7, 1, 6)$ define both extended colorings of \mathcal{C}' of type $\mathcal{C} = (6, 9, 12)$. These colorings are, respectively, represented in the appendix by the factorization of table 16 where we have to consider the following coloring classes $X_1 = \{1, 2, 14, 15, 16, 17\}$, $X_2 = \{3, 4, 5, 6, 7, 18, 19, 20, 21\}$ and $X_3 = \{8, 9, 10, 11, 12, 13, 22, 23, 24, 25, 26, 27\}$, and by the factorization of table 17 with coloring classes $X_1 = \{1, 2, 14, 15, 16, 17, 18, 19, 20\}$, $X_2 = \{3, 4, 5, 6, 7, 21\}$ and $X_3 = \{8, 9, 10, 11, 12, 13, 22, 23, 24, 25, 26, 27\}$.

If a BSTS(27) has an extended 3-coloring with respect to \mathcal{C}' then it is also 4-colorable with the coloring $\mathcal{C}'' = (2, 5, 6, 14)$. Suppose that there exists a 5-coloring for a BSTS(27), by theorems 1.1 and 1.2 for the five coloring classes we have that $n_i \geq 2^{i-1}$ for $1 \leq i \leq 5$. But $\sum_{i=1}^5 n_i > 27$, and this is not possible. Therefore every colorable BSTS(27) cannot be 5-colorable, and a BSTS(27) with an extended 3-coloring has $\chi = 3$ and $\bar{\chi} = 4$. \square

The BSTS(27) is the smallest BSTS which can have an extended coloring. The next two BSTSs obtained by doubling plus one construction are BSTSs(31) and BSTSs(39).

In [8], it was proved that BSTSs(31) have not extended colorings of $\mathcal{C}' = (1, 2, 2^2, 2^3)$. The BSTSs(39) obtained by doubling plus one construction from a BSTS(19) can have extended coloring. In [11] it was proved that a BSTS(19) can be colored just with the following colorings $\mathcal{C}'_1 = (4, 6, 9)$, $\mathcal{C}'_2 = (1, 2, 8, 8)$ and $\mathcal{C}'_3 = (1, 4, 4, 10)$.

Theorem 3.2 *There are colorable BSTSs(39), obtained by doubling plus one construction from a BSTS(19), with extended coloring of $\mathcal{C}'_1 = (4, 6, 9)$ and $\mathcal{C}'_2 = (1, 2, 8, 8)$. In particular for these systems there are BSTSs(39) with $\chi = 3$ and $\bar{\chi} = 4$, with $\chi = 4$ and $\bar{\chi} = 5$ and with $\chi = 3$ and $\bar{\chi} = 5$.*

Proof.

In [11] it was proved that there exist BSTSs(19) uniquely 3-colorable with the coloring $\mathcal{C}'_1 = (4, 6, 9)$, BSTSs(19) uniquely 4-colorable with the coloring $\mathcal{C}'_2 = (1, 2, 8, 8)$ and BSTSs(19) 3 and 4-colorable with the colorings $\mathcal{C}'_1 = (4, 6, 9)$ and $\mathcal{C}'_2 = (1, 2, 8, 8)$, then we have that $\bar{\chi} < 5$ for the colorable BSTSs(19).

Since in \mathcal{C}'_1 there are two n'_1 and n'_2 with $n'_1 + n'_2 = 4 + 6 = (v + 1)/2$ and in \mathcal{C}'_2 two n'_2 e n'_i , where $i = 3$ or 4 , with $n'_1 + n'_i = 2 + 8 = (v + 1)/2$, then by corollary 2.2 we have that for BSTSs(39) there are extended 3-colorings of \mathcal{C}'_1 and of \mathcal{C}'_2 .

The BSTSs(39), obtained by doubling plus one construction from a uniquely 3-colorable BSTS(19), can be colored with the extended coloring $\mathcal{C}_1 = (9, 14, 16)$ and with the 4-coloring $\mathcal{C}_3 = (4, 6, 9, 20)$. These systems are not 5-colorable because their subsystem BSTS(19) is uniquely 3-colorable, therefore $\chi = 3$ and $\bar{\chi} = 4$.

The BSTSs(39), obtained by doubling plus one construction from a uniquely colorable BSTS(19) with \mathcal{C}'_2 , can be colored with the extended coloring $\mathcal{C}_2 = (1, 12, 8, 18)$ and with the 5-coloring $\mathcal{C}_4 = (1, 2, 8, 8, 20)$. These systems cannot be 3-colorable because their subsystem BSTS(19) is uniquely 4-colorable, and $\chi = 4$ e $\bar{\chi} = 5$.

Finally, the BSTSs(39), obtained by doubling plus one construction from a colorable BSTS(19) either with \mathcal{C}'_1 or with \mathcal{C}'_2 , can be 3, 4 and 5-colorable with the extended coloring $\mathcal{C}_1 = (9, 14, 16)$, with the coloring $\mathcal{C}_3 = (4, 6, 9, 20)$ and with the coloring $\mathcal{C}_4 = (1, 2, 8, 8, 20)$, and in this last case $\chi = 3$ and $\bar{\chi} = 5$. \square

BSTS(43) is the next systems which can be obtained by doubling plus one construction from a BSTS(21). A BSTS(21) can be uniquely 3-colorable with one of the two colorings: $\mathcal{C}'_1 = (5, 6, 10)$ and $\mathcal{C}'_2 = (4, 8, 9)$ (see [11]).

All the solutions of the system (1) with respect to \mathcal{C}'_1 and \mathcal{C}'_2 are in the tables 5 and 6, respectively.

c_1	c_2	c_3	c_1	c_2	c_3
5	10	7	6	8	8
5	11	6	6	9	7
11	4	7	12	2	8
11	11	0	12	9	1
12	4	6	13	2	7
12	10	0	13	8	1

table 5

table 6

Theorem 3.3 *The BSTSs(43), (X, \mathcal{B}) , obtained by doubling plus one construction from a colorable BSTS(21), (X', \mathcal{B}') either with $\mathcal{C}'_1 = (5, 6, 10)$ or with $\mathcal{C}'_2 = (4, 8, 9)$, have respectively extended colorings. In particular, for these systems it is $\chi = 3$ and $\bar{\chi} = 4$.*

Proof.

Table 5 shows all the solutions with respect to \mathcal{C}'_1 . In this table, from corollary 2.1, the last three solutions cannot define extended colorings and, from proposition 2.3, the third solution does not define extended colorings. The first and the second solutions of table 5 define extended colorings as we can see in the appendix, tables 18 and 19, respectively. In these tables we have the coloring classes $X_1 = \{1, 2, 3, 4, 5, 22, 23, 24, 25, 26\}$, $X_2 = \{6, 7, 8, 9, 10, 11, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36\}$, and $X_3 = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 37, 38, 39, 40, 41, 42, 43\}$ for the coloring connected with the first solution, and the coloring classes $X_1 = \{1, 2, 3, 4, 5, 22, 23, 24, 25, 26\}$, $X_2 = \{6, 7, 8, 9, 10, 11, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37\}$ and $X_3 = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 38, 39, 40, 41, 42, 43\}$ for the coloring connected with the second solution.

In table 6, that defines solutions with respect to \mathcal{C}'_2 , using corollary 2.1 we have that the last four solutions cannot define extended colorings. The first and the second solutions define extended colorings as we can see in tables 20 and 21 of the appendix. The extended coloring of \mathcal{C}'_2 connected to the first solution has as coloring classes the following ones $X_1 = \{1, 2, 3, 4, 22, 23, 24, 25, 26, 27\}$, $X_2 = \{5, 6, 7, 8, 9, 10, 11, 12, 28, 29, 30, 31, 32, 33, 34, 35\}$, $X_3 = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 36, 37, 38, 39, 40, 41, 42, 43\}$, while the coloring classes of the extended coloring connected to the second solution are $X_1 = \{1, 2, 3, 4, 22, 23, 24, 25, 26, 27\}$, $X_2 = \{5, 6, 7, 8, 9, 10, 11, 12, 28, 29, 30, 31, 32, 33, 34, 35, 36\}$, $X_3 = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 37, 38, 39, 40, 41, 42, 43\}$.

In [9] it was proved that for BSTSs(43), it is $\bar{\chi} \leq 4$, and hence, the theorem is proved. \square

The systems BSTS(51), obtained from doubling plus one construction from a BSTS(25), have not extended colorings of the unique colorings $\mathcal{C}'_1 = (5, 10, 10)$ and $\mathcal{C}'_2 = (1, 4, 8, 12)$ of colorable BSTSs(25) (see [3, 9]).

c_1	c_2	c_3	c_4
0	0	0	26
0	0	6	20
0	16	6	4
0	22	0	4
1	0	8	17
1	13	8	4
3	13	8	2
3	16	6	1
3	22	0	1
16	0	8	2
19	0	6	1
25	0	0	1

table 11

Theorem 3.4 *The systems BSTSs(51), (X, \mathcal{B}) , obtained by doubling plus one construction from a colorable BSTS(25), (X', \mathcal{B}') with $\mathcal{C}'_1 = (5, 10, 10)$ or $\mathcal{C}'_2 = (1, 4, 8, 12)$, have not extended colorings.*

Proof.

Table 11 shows all the solutions with respect to \mathcal{C}'_1 . From corollary 2.1 and proposition 2.2, none of them defines an extended coloring. There are not solutions of the system (1) with respect to \mathcal{C}'_2 , so the theorem follows. \square

BSTS($2v + 1$)	BSTS(55)	BSTS(79)	BSTS(87)
BSTS(v)	BSTS(27)	BSTS(39)	BSTS(43)
\mathcal{C}'_1	(1,4,10,12)	(1,8,12,18)	(1,10,12,20)
\mathcal{C}'_2	(2,5,6,14)	(2,6,13,18)	(4,4,17,18)
\mathcal{C}'_3	–	(4,6,9,20)	–
\mathcal{C}'_4	–	(1,2,8,8,20)	–
\mathcal{C}'_5	–	(1,4,4,10,20)	–

table 12

The next BSTSs obtained by doubling plus one construction are BSTSs(55), BSTSs(79) and BSTSs(87), and they are, respectively, constructed from BSTSs(27), BSTSs(39) and BSTSs(43). In table 12, by theorem 2.2, we can find colorings \mathcal{C}' of BSTSs(27), BSTSs(39) and BSTSs(43) that are extendible, i.e., colorings which define extended colorings. The colorings \mathcal{C}' are obtained using the conditions of theorem 1.3.

For systems BSTSs(63) obtained by doubling plus one construction from BSTSs(31), we know, from [8], that they have not extended colorings. Finally, the systems BSTSs(67) and BSTSs(75), obtained from the systems BSTSs(33) and BSTSs(37), respectively, do not admit extended colorings because none of the coloring \mathcal{C}' of the systems BSTSs(33) and BSTSs(37), gives solutions of system (1) (see [9]).

Theorem 3.5 *The systems BSTSs(55) and BSTSs(87) have extended 4-coloring, the systems BSTSs(79) have extended 4 and 5-colorings. The systems BSTSs(63), BSTSs(67), BSTSs(75) have not extended colorings. \square*

The systems BSTSs(91), obtained by doubling plus one construction from a BSTS(45), can have extended colorings with respect to the colorings $\mathcal{C}'_1 = (2, 8, 14, 21)$ and $\mathcal{C}'_2 = (4, 6, 13, 22)$. The colorings \mathcal{C}'_1 and \mathcal{C}'_2 are the unique colorings that verify the hypotheses of theorem 1.3 and in [3] it is proved their existence. In the tables 13 and 14 there are all the solutions with respect to \mathcal{C}'_1 and \mathcal{C}'_2 .

c_1	c_2	c_3	c_4
6	6	11	23
3	6	20	17
3	6	24	13
3	26	0	17
3	30	0	13
6	6	30	4
6	17	0	23
6	36	0	4
12	0	11	23
12	0	30	4
23	0	0	23
42	0	0	4

table 13

c_1	c_2	c_3	c_4
4	8	12	22
10	2	12	22
1	8	21	16
1	8	25	12
1	28	1	16
1	32	1	12
4	8	31	3
4	19	1	22
4	38	1	3
10	2	31	3
21	2	1	22
40	2	1	3

table 14

Theorem 3.6 *The systems $BSTSs(91)$, (X, \mathcal{B}) , obtained by doubling plus one construction from a $BSTS(45)$, (X', \mathcal{B}') , have extended 4-colorings of \mathcal{C}_2 .*

Proof.

In the table 13 there are all the solutions with respect to \mathcal{C}'_1 . From corollary 2.1 and propositions 2.2 and 2.3, we have that these solutions do not define extended colorings of \mathcal{C}'_1 .

In the table 14 there are all the solutions with respect to \mathcal{C}'_2 . The first solution defines an extended 4-coloring which is obtained by the factorization of table 22 with the following coloring classes $X_1 = \{1, 2, 3, 4, 46, 47, 48, 49\}$, $X_2 = \{5, 6, 7, 8, 9, 10, 50, 51, 52, 53, 54, 55, 56, 57\}$, $X_3 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69\}$, $X_4 = \{24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91\}$.

Suppose that the second solution of table 14 permits a factorization \mathcal{F} which defines an extended h -coloring of \mathcal{C}'_2 . We have that all the 24 non monochromatic pairs of vertices of X'' colored with the colors [2] and [3] have to be in 19 factors corresponding to the vertices $x_l \in X'$ colored with [2] and [3]. The two vertices of X'' colored with [2] define one monochromatic pair which is in only one factor of \mathcal{F} . Moreover, in all of the other factors, these two vertices have to be always in two different non monochromatic pairs. So 24 non monochromatic pairs of colors [2] and [3] are in 12 factors corresponding to the vertices $x_l \in X'$ of color [2] and [3]. Finally, if we consider at least six factors containing the non monochromatic pairs of colors [2] and [i] with $i \neq 3$, then this solution cannot define an extended coloring.

The remaining solutions of table 14, by corollary 2.1 and propositions 2.2 and 2.3, do not permit extended coloring of \mathcal{C}'_2 . \square

It remains to study the BSTSs(99) obtained by doubling plus one construction from a BSTS(49). A BSTS(49) can be colored with just five colorings: $\mathcal{C}'_1 = (12, 17, 20)$, $\mathcal{C}'_2 = (14, 14, 21)$, $\mathcal{C}'_3 = (1, 4, 4, 20, 20)$, $\mathcal{C}'_4 = (5, 6, 14, 24)$ and $\mathcal{C}'_5 = (2, 8, 18, 21)$. All these colorings verify the hypothesis of theorem 1.3 and their existence is proved in [3].

Theorem 3.7 *The BSTSs(99), (X, \mathcal{B}) , obtained by doubling plus one construction from a BSTSs(49), (X', \mathcal{B}') , have extended 4 and 5-colorings.*

Proof.

To simplify the proof, we will not list all the solutions of the system (1) with respect to the colorings \mathcal{C}'_1 , \mathcal{C}'_2 , \mathcal{C}'_3 , \mathcal{C}'_4 and \mathcal{C}'_5 , and we will also omit all the solutions which do not respect the necessary conditions defined in the previous section.

The colorings \mathcal{C}'_1 and \mathcal{C}'_2 do not admit solutions for the system (1), so BSTSs(99) have not extended 3-colorings.

There are 84 solutions of the system (1) with respect to \mathcal{C}'_3 , but from corollary 2.1 and propositions 2.2 and 2.3 only three among all the solutions are significative. They are: $(4, 1, 5, 19, 21)$, $(3, 2, 5, 20, 20)$ e $(0, 0, 12, 16, 22)$.

The solution $(4, 1, 5, 19, 21)$ cannot define an extended 5-coloring because this coloring would have more than one coloring class with odd cardinality, and this is not possible using theorem 8 of [11].

The solution $(3, 2, 5, 20, 20)$ does not admit an extended coloring because in a factorization of the vertices X'' it is not possible to find in the four factors corresponding to the vertices $x_l \in X'$ of color [2] all the three monochromatic pairs of color [1] with respect to the six non monochromatic pairs of colors [1] and [2].

The solution $(0, 0, 12, 16, 22)$ admits an extended 5-coloring defined by the factorization of vertices X'' in table 23 of the appendix. The coloring classes of this coloring are $X_1 = \{1\}$, $X_2 = \{2, 3, 4, 5\}$, $X_3 = \{6, 7, 8, 9, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61\}$, $X_4 = \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77\}$ e $X_5 = \{30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$.

There are 29 solutions of the system (1) with respect to \mathcal{C}'_4 , but only three of them are significative: $(5, 10, 10, 25)$, $(11, 4, 10, 25)$ and $(3, 11, 12, 24)$.

Suppose that the solution $(5, 10, 10, 25)$ defines an extended coloring. The 50 non monochromatic pairs of vertices of X'' , colored with [1] and [2], have necessarily to be in the 11 factors corresponding to the vertices $x_l \in X'$ of colors [1] and [2] in the following way: 4 of these pairs have to be in 5 factors corresponding to the vertices $x_l \in X'$ of color [1] and 5 pairs in 6 factors tied to the vertices $x_l \in X'$ of color [2]. Necessarily the 300 monochromatic pairs of vertices of X'' of color [4] have to be in 25 factors corresponding to the vertices $x_l \in X'$ of colors [1], [2] and [3], exactly in 12 pairs for every factor. This disposition obliges to put only 43 monochromatic pairs of color [3] on the 45 factors corresponding to the vertices $x_l \in X'$ of colors [1] and [2], and this does not permit a correct extended coloring.

Suppose that the solution $(11, 4, 10, 25)$ determines an extended coloring. The 44 non monochromatic pairs of the vertices of X'' , colored with [1] and [2], have necessarily to be in 11 factors corresponding to the vertices $x_l \in X'$ of colors [1] and [2] in following way: 4 of these pairs have to be in 5 factors corresponding with the vertices $x_l \in X'$ of color [1] and only 3 pairs have to be in 6 factors corresponding with the vertices $x_l \in X'$ of color [2]. The number of these pairs is 38, while all these pairs are 44, and this does not permit a correct extended coloring.

The solution $(3, 11, 12, 24)$ gives an extended 4-coloring which is defined by the factorization of the vertices of X'' in the table 24 of the appendix. This extended coloring has the following coloring classes $X_1 = \{1, 2, 3, 4, 5, 50, 51, 52\}$, $X_2 = \{6, 7, 8, 9, 10, 11, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63\}$, $X_3 = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75\}$, $X_4 = \{26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$.

There are 27 solutions of the system (1) with respect to \mathcal{C}'_5 , but from corollary 2.1 and propositions 2.2 and 2.3, we have that only three of them are significative: $(8, 2, 19, 21)$, $(6, 4, 20, 20)$ and $(10, 0, 20, 20)$.

The solution $(8, 2, 19, 21)$ does not defines extended colorings because in X'' there are two vertices of color [2] and a odd number of vertices of colors [3] and [4].

The solution $(6, 4, 20, 20)$ defines an extended 4-coloring by the factorization of X'' represented on table 25. This extended coloring has the following coloring classes $X_1 = \{1, 2, 50, 51, 52, 53, 54, 55\}$, $X_2 = \{3, 4, 5, 6, 7, 8, 9, 10, 56, 57, 58, 59\}$, $X_3 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79\}$, $X_4 = \{29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$.

The solution $(10, 0, 20, 20)$ defines an other extended 4-coloring represented by the factorization on the table 26 and having the following coloring classes $X_1 = \{1, 2, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59\}$, $X_2 = \{3, 4, 5, 6, 7, 8, 9, 10\}$, $X_3 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79\}$, $X_4 = \{29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$. \square

4 Extended colorings for BSTSs($2^h - 1$)

The problem connected to the determination of an extended colorings for the particular systems of type BSTS($2^h - 1$) was studied in [8], where the authors found necessary conditions for the existence of extended colorings. They also proved that extended colorings do not exist when $h < 10$. The problem to understand if there are extended colorings for $h \geq 10$ remains still open.

In the following theorems \bar{h} denotes the maximum value of h for which the system BSTS($2^{\bar{h}} - 1$) has not extended coloring.

Theorem 4.1 ([8]) *If \mathcal{C} be an extended \bar{h} -coloring of a BSTS($2^{\bar{h}+1} - 1$), then there exists at least one $c_i = 0$.* \square

Theorem 4.2 ([8]) *Let \mathcal{C} be an extended \bar{h} -coloring of a BSTS($2^{\bar{h}+1} - 1$), $c_i > 0$ and $c_l = 0$ for $l < i$ ($i = 1$ is possible), and let $c_j > 0$ ($j = i + 1$ is possible) and $c_k = 0$ for $i + 1 \leq k < j$. Then $c_{j+t} > 0$ for all $j + t > i + 1$.* \square

Notice that if there exists an extended \bar{h} -coloring of a BSTS($2^{\bar{h}+1} - 1$), then, by corollary 2.1, the connected solution of the system (1) necessarily has all the $c_i > 0$ even.

The following propositions permits to prove that the unique solutions of systems (1), which could define an extended colorings for BSTSs($2^h - 1$), are only the solutions defined in theorem 4.2, but with $c_1 = 0$.

Proposition 4.1 *There are not extended \bar{h} -colorings of a BSTS($2^{\bar{h}+1} - 1$) corresponding to a solution of type $(0, \dots, 0, c_i, c_{i+1}, \dots, c_{i+c}, 0, \dots, 0, c_j, \dots, c_{\bar{h}})$, where $c_p > 0$ for $i \leq p \leq i + c$ ($i = 1$ is possible) with $c \geq 1$ and $j \leq p \leq \bar{h}$.*

Proof.

Suppose that \mathcal{C} is an extended \bar{h} -coloring corresponding to a solution of the system (1) of type $(0, \dots, 0, c_i, c_{i+1}, \dots, c_{i+c}, 0, \dots, 0, c_j, \dots, c_{\bar{h}})$. It defines a factorization \mathcal{F} of the vertices in X'' . If $x' \in X''$ is a vertex colored with the color $[i]$ in \mathcal{C} , then it is in a monochromatic pair of color $[i]$ for every factor \mathcal{F} corresponding to the vertices $x_l \in X'$ of color $[q]$, where $1 \leq q \leq i-1$ and $i+c+1 \leq q \leq j-1$. From proposition 2.2, we have

$$2^{j-1} - 2^{i+c} + 2^{i-1} \leq c_i \leq 2^{i-1} + 2^i,$$

from which it follows

$$2^{j-1} \leq 2^{i+c} + 2^i = 2^c \cdot 2^i + 2^i = 2^i \cdot (2^c + 1).$$

If we define $r = j - (i + c)$, then we have

$$2^i \cdot 2^{c+r-1} \leq 2^i \cdot (2^c + 1).$$

Let us consider $r \geq 2$,

$$2^{c+1} \leq 2^{c+r-1} \leq 2^c + 1.$$

These last inequality is equivalent to $2^c \leq 1$, this is impossible because $c \geq 1$. Consider the case $r = 1$, we have a solution of type $(0, \dots, 0, c_i, c_{i+1}, \dots, c_{j-2}, 0, c_j, \dots, c_{\bar{h}})$ where $j \geq i + 3$ because $r = 1$. We have the following inequalities

$$2^{j-1} - 2^{j-2} + 2^{i-1} \leq c_i \leq 2^{i-1} + 2^i,$$

so we obtain that $2^{j-2} \leq 2^i$, and it is true only if $j < i + 3$, but this is not possible because $j \geq i + 3$ and theorem is proved. \square

Proposition 4.2 *If \mathcal{C} is an extended \bar{h} -coloring of a BSTS($2^{\bar{h}+1} - 1$), then $c_1 = 0$.*

Proof.

From proposition 4.1 and theorem 4.2, we have that the unique possible extended colorings with $c_1 \neq 0$ can be connected to the solutions of the system (1) of type $(c_1, 0, \dots, 0, c_j, c_{j+1}, \dots, c_{\bar{h}})$. Suppose that this solution defines an extended coloring \mathcal{C} .

From proposition 2.2, it is $c_1 \leq 2^{j-1} + 1$, but c_1 has to be even, so $c_1 \leq 2^{j-1}$. Let $x' \in X''$ be a vertex which is colored with the color $[1]$ in \mathcal{C} . In the

factorization \mathcal{F} of X'' , defined by \mathcal{C} , x' is in one non monochromatic pair of colors $[1]$ and $[p]$, which is contained in the unique factor corresponding to the vertex $x_1 \in X'$ of color $[1]$, and in $2^{j-1} - 2$ monochromatic pairs colored with $[1]$ and all of them are contained in the factors corresponding to the vertices $x_l \in X'$ colored with the colors $[k]$, for $2 \leq k \leq j - 1$. Since c_1 has to be even, then $c_1 \geq 2^{j-1}$, i.e., $c_1 = 2^{j-1}$.

Let us consider one $c_{k'}$, with $k' \geq j$, from proposition 2.2 it has to be even, therefore we have that $c_{k'} \leq 2^{k'-1}$. Since $c_1 = 2^{j-1}$, then $x' \in X''$ of color $[1]$ can be in $2^{j-1} - 1$ monochromatic pairs of color $[1]$ where, in particular, $2^{j-1} - 2$ pairs are in the factors corresponding to the vertices $x_l \in X'$ colored with the colors $[k]$ with $2 \leq k \leq j - 1$ and only one is in a factor corresponding to a vertex $x_l \in X'$ colored with a color $[k']$ where $j \leq k' \leq \bar{h}$.

Therefore in all the factors corresponding to the vertices $x_l \in X'$ of color $[k']$, with $j \leq k' \leq \bar{h}$, x' is contained in at least $2^{k'-1} - 1$ different pairs colored with the colors $[1]$ and $[k']$, so we have $c_{k'} \geq 2^{k'-1} - 1$ and because $c_{k'}$ has to be even, then $c_{k'} = 2^{k'-1}$ for $j \leq k' \leq \bar{h}$. If we replace these values in the equations of system (1), we have:

$$\begin{cases} 2^{2j-2} + \sum_{k'=j}^{\bar{h}} 2^{2k'-2} + 2^j + \sum_{k'=j}^{\bar{h}} 2^{2k'-1} = 2^{2\bar{h}} \\ \sum_{i=1}^h c_i = 2^{\bar{h}}. \end{cases} \quad (2)$$

It is easy to check that the second equation of system (2) is true, while the first one is not verified because $2^j \neq 0$. Thus, the solution $(c_1, 0, \dots, 0, c_j, c_{j+1}, \dots, c_{\bar{h}})$ does not permit to define a extended coloring. \square

From the previous two propositions we have that the unique solutions of the system (1) defining extended colorings are only the solutions of theorem 4.2 with $c_1 \neq 0$. We can classify them in the following two typology:

1. $(0, \dots, 0, c_i, 0, \dots, 0, c_j, c_{j+1}, \dots, c_{\bar{h}})$;
2. $(0, \dots, 0, c_i, c_{i+1}, \dots, c_{\bar{h}})$.

The following theorem permits to determine an infinite class of BSTSs($2^h + 1$) with not extended colorings for any integer h .

Theorem 4.3 *For every $h \geq 1$ there exists a BSTS($2^h - 1$) which has not extended colorings.*

Proof. As it was proved in [8], there exists certainly a $\bar{h} > 1$ for which there is a $S' = \text{BSTS}(2^{\bar{h}} - 1)$, (X', \mathcal{B}') with not extended colorings. We are going to show that it is always possible to determine a $S = \text{BSTS}(2^{\bar{h}+1} - 1)$, (X, \mathcal{B}) , obtained by doubling plus one construction from S' , which does not admit extended colorings.

Let \mathcal{C}' be a \bar{h} -coloring of S' , we define a factorization \mathcal{F} of the vertex set X'' , with $X' \cap X'' = \emptyset$ and $|X''| = 2^{\bar{h}}$, which does not permit an extended coloring of the system S . If we consider $X''_h \subset X''$, with $|X''_h| = 2^{\bar{h}-1}$, then we can easily built $2^{\bar{h}-1}$ factors of \mathcal{F} such that in all of these factors there are not pairs containing two vertices of X''_h .

Let $\mathcal{F}^{(1)}$ be a factorization of the vertices of X''_h , it has $2^{\bar{h}-1} - 1$ factors, and every factor contains $2^{\bar{h}-2}$ pairs. Let $\mathcal{F}^{(2)}$ be the factorization of vertices $X'' - X''_h$ where $|X'' - X''_h| = 2^{\bar{h}-1}$.

If we label the factors of $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ and consider the k -th factor in $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ then the union of all the $2^{\bar{h}-2}$ pairs in these two factors defines a factor of \mathcal{F} . If we repeat this procedure for every k with $1 \leq k \leq 2^{\bar{h}-1} - 1$, then we obtain $2^{\bar{h}-1} - 1$ factors that together with the previous $2^{\bar{h}-1}$ factors define all the complete factorization \mathcal{F} .

Finally we obtain the system $S = \text{BSTS}(2^{\bar{h}+1} - 1)$, by doubling plus one construction from S' , in the following way: the $2^{\bar{h}-1} - 1$ factors, obtained from the union of the pairs in the factors of $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$, are connected with the vertices $x_i \in X'$ colored with the colors $[j]$ with $1 \leq j \leq \bar{h} - 1$ in \mathcal{C}' , while all the other factors of \mathcal{F} are connected with the vertices $x_i \in X'$ of color $[\bar{h}]$ in \mathcal{C}' .

In this system, because S' has not extended coloring we have that the pairs in $\mathcal{F}^{(2)}$ cannot be colored with any color $[i]$, for $1 \leq i \leq \bar{h} - 1$, of \mathcal{C}' and the system S has not extended coloring. It is clear that recursively, for every integer h , it is always possible to obtain a $\text{BSTS}(2^h - 1)$ which has not extended colorings and the theorem follows. \square

5 Concluding remarks

In this paper, we determine particular strict colorings, called extended colorings, of particular $\text{BSTS}(2v+1)$, obtained by doubling plus one construction from a h -colorable $\text{BSTS}(v)$. If \mathcal{C}' is an edge coloring of $\text{BSTS}(v)$, then the $\text{BSTS}(2v+1)$ has an extended coloring \mathcal{C} of \mathcal{C}' if the subsystem $\text{BSTS}(v)$ is colored with \mathcal{C}' .

We prove that the unique systems $BSTSs(v)$, for $v < 103$, with extended colorings are the systems whose order are in the following set $V = \{27, 39, 43, 55, 79, 87, 91, 99\}$. The table 15 resumes all the results obtained in section 3, i.e., it contains all the $BSTSs$, of order less than 103, which have extended h -coloring for all the possible h .

If a $BSTS(v)$ can be colored with an extended coloring, then we have that the upper and lower chromatic number have distinct values, i.e., $\chi \neq \bar{\chi}$. The existence of these particular colorings gives a useful tool to study all the possible different kind of feseable sets for a single colorable $BSTS(v)$, where a feseable set of a $BSTS(v)$ is the set $\Omega(v) = \{k : \text{for which there exists a } k\text{-coloring}\}$ (see [9]). In particular, all the $BSTSs(v)$ with $v \in V$ and with extended coloring permit to define infinite classes of $BSTSs$ which have two strict colorings with a different number of colors. Theorem 5.1 gives this property and it can be proved easily repeating a sequence of doubling plus one constructions.

$BSTS(v)$	3-colorings	4-colorings	5-colorings
$BSTS(27)$	yes	no	no
$BSTS(39)$	yes	yes	no
$BSTS(43)$	yes	no	no
$BSTS(55)$	no	yes	no
$BSTS(79)$	no	yes	yes
$BSTS(87)$	no	yes	no
$BSTS(91)$	no	yes	no
$BSTS(99)$	no	yes	yes

table 15

Theorem 5.1 *If a $BSTS(v)$ with $v \in \{27, 39, 43, 55, 79, 87, 91, 99\}$ has an extended h -colorings, then there exists an infinite class of $BSTSs(2^k(v+1) - 1)$ with at least two $(h+k-1)$ e $(h+k)$ -colorings, and in particular we have that $\chi \neq \bar{\chi}$. \square*

For the systems of type $BSTSs(2^h - 1)$ we give new necessary and sufficient conditions for finding extending colorings. In particular, we obtain that the solutions of the system (1) which define extended colorings can only be of two types: $(0, \dots, 0, c_i, 0, \dots, 0, c_j, c_{j+1}, \dots, c_{\bar{h}})$ and $(0, \dots, 0, c_i, c_{i+1}, \dots, c_{\bar{h}})$. It is important to underline that extended coloring have been finding for these particular systems and the following conjecture is still true [7].

Conjecture 5.1 ([7]) *If a $BSTS(2^h - 1)$ is obtained by a sequence of doubles plus one constructions from $BSTS(3)$, then it has not extended colorings for every $h \geq 1$.*

6 Appendix

In this appendix we list all the factorizations \mathcal{F} which characterize all the extended colorings shown in section 3 of this paper. In the following, until table 21, we will insert the coloring classes before of the factorizations \mathcal{F} . All the coloring classes for the tables 22, 23, 24, 25 and 26 are defined at page 31.

BSTS(27)

Coloring classes: $X_1 = \{1, 2, 14, 15, 16, 17\}$, $X_2 = \{3, 4, 5, 6, 7, 18, 19, 20, 21\}$ and $X_3 = \{8, 9, 10, 11, 12, 13, 22, 23, 24, 25, 26, 27\}$.

1	2	3	4	5	6	7
14 18	14 21	18 16	18 17	18 22	18 23	18 25
15 19	15 18	19 17	19 14	19 23	19 26	19 22
16 20	16 19	14 20	20 15	20 26	20 24	20 27
17 21	17 20	15 21	21 16	21 27	21 25	21 24
22 23	22 24	22 25	22 26	14 15	14 16	14 17
24 27	25 23	24 26	25 27	16 17	15 17	15 16
25 26	26 27	23 27	23 24	24 25	22 27	23 26

8	9	10	11	12	13
22 14	22 15	22 20	22 16	22 17	22 21
23 15	23 16	23 21	23 14	23 20	23 17
24 16	24 17	24 14	24 18	24 19	24 15
25 17	25 14	25 15	25 20	25 16	25 19
26 18	26 21	26 16	26 17	26 15	26 14
27 19	27 18	27 17	27 15	27 14	27 16
20 21	19 20	18 19	19 21	18 21	18 20

table 16

Coloring classes: $X_1 = \{1, 2, 14, 15, 16, 17, 18, 19, 20\}$, $X_2 = \{3, 4, 5, 6, 7, 21\}$ and $X_3 = \{8, 9, 10, 11, 12, 13, 22, 23, 24, 25, 26, 27\}$.

1	2	3	4	5	6	7
14 21	14 23	22 23	22 24	22 25	22 26	22 27
15 27	15 26	24 27	23 25	24 26	25 27	23 26
16 24	16 27	25 26	26 27	23 27	23 24	24 25
17 23	17 25	14 16	15 17	16 18	17 19	18 20
18 25	18 24	17 20	14 18	15 19	16 20	14 17
19 26	19 22	18 19	19 20	14 20	14 15	15 16
20 22	20 21	21 15	21 16	21 17	21 18	21 19

8	9	10	11	12	13
22 21	22 14	22 15	22 16	22 17	22 18
23 18	23 21	23 16	23 15	23 19	23 20
24 14	24 15	24 21	24 17	24 20	24 19
25 16	25 20	25 19	25 21	25 14	25 15
26 17	26 18	26 14	26 20	26 16	26 21
27 19	27 17	27 20	27 18	27 21	27 14
15 20	16 19	17 18	14 19	15 18	16 17

table 17

BSTS(43)

Coloring classes: $X_1 = \{1, 2, 3, 4, 5, 22, 23, 24, 25, 26\}$, $X_2 = \{6, 7, 8, 9, 10, 11, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36\}$, $X_3 = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 37, 38, 39, 40, 41, 42, 43\}$.

1	2	3	4	5	6	7	8	9
22 27	23 29	26 28	26 36	25 33	23 27	24 27	25 27	26 27
23 28	24 31	25 34	25 35	24 34	22 28	25 28	24 32	22 32
25 29	22 35	24 35	24 28	23 35	24 29	26 29	26 33	23 33
26 30	26 34	23 36	22 29	22 36	25 30	22 30	23 30	24 30
24 43	25 39	22 42	23 41	26 40	26 32	23 31	22 31	25 31
31 34	27 36	27 31	27 32	31 32	31 37	32 38	28 39	28 38
32 33	28 32	30 32	31 33	28 29	33 41	35 43	29 42	29 43
35 36	30 33	29 33	30 34	27 30	34 42	33 40	34 37	34 40
37 42	38 40	41 43	40 42	39 41	35 40	34 41	35 38	35 42
38 41	37 41	37 40	39 43	38 42	36 39	36 37	36 43	36 41
39 40	42 43	38 39	37 38	37 43	38 43	39 42	40 41	37 39

10	11	12	13	14	15	16	17	18
23 32	22 33	22 39	22 37	22 43	22 38	22 41	30 37	34 39
24 33	25 32	23 42	23 38	23 39	23 43	23 40	31 38	33 42
22 34	23 34	24 40	24 41	24 42	24 39	24 37	27 39	32 41
26 35	24 36	25 41	25 42	25 37	25 40	25 38	32 43	31 40
25 36	26 31	26 37	26 43	26 38	26 42	26 39	29 41	30 43
28 37	28 40	33 38	31 39	28 41	29 37	30 42	28 42	27 38
29 38	29 39	34 43	32 40	29 40	30 41	31 43	22 40	23 37
30 39	30 38	27 29	27 28	27 33	27 34	27 35	23 26	22 24
31 41	35 37	28 30	29 36	32 34	33 35	34 36	24 25	25 26
27 42	27 43	31 36	30 35	31 35	32 36	28 33	33 36	29 35
40 43	41 42	35 32	33 34	30 36	28 31	29 32	34 35	28 36

19	20	21
36 42	38 36	36 40
35 41	35 39	34 38
33 39	33 37	33 43
32 37	32 42	32 39
28 43	30 40	27 37
27 40	27 41	26 41
24 38	25 43	22 25
23 25	24 26	23 24
22 26	22 23	29 30
30 31	29 31	28 35
29 34	28 34	31 42

table 18

Coloring classes: $X_1 = \{1, 2, 3, 4, 22, 23, 24, 25, 26\}$, $X_2 = \{5, 6, 7, 8, 9, 10, 11, 12, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37\}$, $X_3 = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 38, 39, 40, 41, 42, 43\}$.

1	2	3	4	5	6	7	8	9
22 27	22 37	22 28	27 32	22 29	27 41	27 43	27 38	27 25
23 28	23 30	23 27	23 29	23 31	28 26	28 42	28 39	28 43
24 29	24 28	24 31	24 30	24 32	29 38	29 40	29 41	29 42
25 36	25 29	25 30	25 28	25 33	30 22	30 39	30 40	30 41
26 37	26 27	26 29	26 31	26 30	31 25	31 22	31 42	31 38
38 39	38 40	38 41	38 42	38 43	32 23	32 25	32 43	32 26
40 43	39 41	40 42	41 43	39 42	33 39	33 41	33 24	33 22
41 42	42 43	39 43	39 40	40 41	34 40	34 23	34 22	34 24
30 35	31 36	32 37	27 33	28 34	35 24	35 26	35 23	35 39
31 34	32 35	33 36	34 37	27 35	36 42	36 24	36 26	36 23
32 33	33 34	34 35	35 36	36 37	37 43	37 38	37 25	37 40

10	11	12	13	14	15	16	17	18
27 42	27 24	38 23	38 22	38 26	38 25	38 35	38 24	38 36
28 41	28 38	39 32	39 24	39 34	39 23	39 22	39 25	39 27
29 39	29 43	40 24	40 23	40 22	40 26	40 28	40 36	40 35
30 43	30 42	41 25	41 26	41 24	41 22	41 32	41 23	41 34
31 40	31 39	42 26	42 25	42 23	42 35	42 37	42 22	42 33
32 38	32 40	43 22	43 33	43 25	43 24	43 31	43 26	43 23
33 23	33 26	31 33	32 34	33 35	34 36	27 36	35 37	28 37
34 26	34 25	30 34	31 35	32 36	33 37	29 34	34 27	30 32
35 25	35 22	29 35	30 36	31 37	27 32	30 33	28 33	29 31
36 22	36 41	28 36	29 37	27 30	28 31	23 26	29 32	22 24
37 24	37 23	27 37	27 28	28 29	29 30	24 25	30 31	25 26

19	20	21
38 34	38 33	38 30
39 36	39 37	39 26
40 33	40 25	40 27
41 31	41 35	41 37
42 24	42 32	42 34
43 35	43 34	43 36
27 29	28 30	29 33
30 37	27 31	31 32
28 32	29 36	28 35
23 25	24 26	22 25
22 26	22 23	23 24

table 19

Coloring classes: $X_1 = \{1, 2, 3, 4, 22, 23, 24, 25, 26, 27\}$, $X_2 = \{5, 6, 7, 8, 9, 10, 11, 12, 28, 29, 30, 31, 32, 33, 34, 35\}$, $X_3 = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 36, 37, 38, 39, 40, 41, 42, 43\}$.

1	2	3	4	5	6	7	8	9
22 28	22 29	22 30	22 31	35 36	28 41	28 38	32 42	32 39
23 29	26 28	23 31	23 32	29 38	29 36	29 39	34 43	29 40
24 30	24 31	27 29	24 33	30 39	30 40	30 37	30 36	33 41
25 31	25 32	25 33	25 34	31 40	31 37	31 42	31 38	31 36
26 40	23 41	26 42	26 37	32 22	32 27	32 26	28 23	28 25
27 41	27 42	24 43	27 38	33 23	33 22	33 27	33 26	35 22
32 33	30 34	28 35	28 29	28 27	35 23	35 24	29 24	34 27
34 35	33 35	32 34	30 35	34 26	34 24	34 22	35 25	30 26
36 43	36 38	36 39	36 41	37 43	38 43	36 40	37 40	37 38
37 42	37 39	38 40	40 42	41 42	39 42	41 43	39 41	42 43
38 39	40 43	37 41	39 43	24 25	25 26	23 25	22 27	23 24

10	11	12	13	14	15	16	17	18
28 39	32 41	28 42	28 36	28 37	28 40	28 43	29 41	29 42
29 43	34 37	34 40	29 37	35 39	35 41	33 39	35 37	32 38
32 37	35 43	33 43	35 38	34 38	34 36	35 40	33 40	33 37
33 43	28 24	31 39	34 39	32 40	32 43	31 41	30 42	30 43
30 25	29 25	35 27	23 40	22 36	27 37	26 36	27 43	27 40
31 27	30 27	29 26	24 42	24 41	25 38	22 37	25 39	25 36
35 26	31 26	32 24	26 41	23 42	23 39	24 38	24 36	24 39
34 23	36 42	30 23	22 43	25 43	22 42	25 42	23 38	22 41
40 41	39 40	36 37	25 27	26 27	24 26	23 27	22 26	23 26
38 42	22 23	38 41	30 33	29 33	29 30	29 34	28 32	28 34
22 24	33 38	22 25	31 32	30 31	31 32	30 32	31 34	31 35

19	20	21
30 41	33 42	35 42
32 36	34 41	30 38
34 42	36 23	36 27
31 43	37 25	37 24
26 38	38 22	39 26
25 40	39 27	40 22
23 37	40 24	41 25
22 39	43 26	43 23
24 27	29 31	28 31
29 35	28 30	33 34
28 33	32 35	29 32

table 20

Coloring classes: $X_1 = \{1, 2, 3, 4, 22, 23, 24, 25, 26, 27\}$, $X_2 = \{5, 6, 7, 8, 9, 10, 11, 12, 28, 29, 30, 31, 32, 33, 34, 35, 36\}$, $X_3 = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 37, 38, 39, 40, 41, 42, 43\}$

1	2	3	4	5	6	7	8	9
22 30	27 28	24 35	26 36	28 23	36 22	28 22	29 22	31 22
23 34	23 38	22 37	22 40	29 24	33 23	30 23	31 23	36 23
25 32	24 40	23 41	23 42	30 25	32 24	34 24	30 24	33 24
24 41	25 39	25 42	24 43	31 26	34 25	36 25	35 25	29 25
26 37	26 42	26 43	25 37	32 27	28 26	29 26	34 26	35 26
27 40	22 43	27 38	27 39	33 22	31 27	35 27	33 27	34 27
38 43	37 41	39 40	38 41	34 37	29 43	31 38	28 37	28 41
39 42	29 36	28 30	28 33	35 40	30 37	32 39	32 41	30 39
29 31	30 35	29 33	29 34	36 42	35 42	33 43	36 38	32 38
33 36	31 34	31 36	30 32	38 39	39 41	37 40	39 43	37 42
28 35	32 33	32 34	31 35	41 43	38 40	41 42	40 42	40 43

10	11	12	13	14	15	16	17	18
32 22	34 22	35 22	37 29	37 31	37 32	37 35	37 36	38 22
35 23	32 23	29 23	38 30	38 28	38 33	38 34	38 35	37 23
28 24	36 24	31 24	39 31	39 33	39 35	39 29	39 34	42 24
33 25	31 25	28 25	40 32	40 34	40 36	40 28	40 33	41 25
30 26	33 26	32 26	41 33	41 36	41 29	41 30	41 31	39 26
36 27	29 27	30 27	42 34	42 30	42 28	42 32	42 29	43 27
29 38	28 39	36 39	43 35	43 32	43 31	43 36	43 30	29 40
31 40	30 40	33 42	22 23	22 24	22 25	22 26	22 27	28 31
34 41	35 41	34 43	24 27	23 25	24 26	25 27	23 26	30 36
37 39	38 42	37 38	25 26	26 27	23 27	23 24	24 25	32 35
42 43	37 43	40 41	28 36	29 35	30 34	31 33	28 32	33 34

19	20	21
42 22	41 22	39 22
39 23	43 23	40 23
37 24	39 24	38 24
40 25	38 25	43 25
38 26	40 26	41 26
41 27	42 27	37 27
28 43	33 37	31 42
29 30	28 34	28 29
31 32	29 32	30 33
33 35	30 31	34 35
34 36	35 36	32 36

table 21

BSTS(91)

Coloring classes of Table 22: $X_1 = \{1, 2, 3, 4, 46, 47, 48, 49\}$, $X_2 = \{5, 6, 7, 8, 9, 10, 50, 51, 52, 53, 54, 55, 56, 57\}$, $X_3 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69\}$, $X_4 = \{24, 25, 26, 27, 28, 29, 30, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 45, 70, 71, 45, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91\}$.

BSTS(99)

Coloring classes of Table 23: $X_1 = \{1\}$, $X_2 = \{2, 3, 4, 5\}$, $X_3 = \{6, 7, 8, 9, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61\}$, $X_4 = \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77\}$ e $X_5 = \{30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$.

Coloring classes of Table 24: $X_1 = \{1, 2, 3, 4, 5, 50, 51, 52\}$, $X_2 = \{6, 7, 8, 9, 10, 11, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63\}$, $X_3 = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75\}$, $X_4 = \{26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$.

Coloring classes of Table 25: $X_1 = \{1, 2, 50, 51, 52, 53, 54, 55\}$, $X_2 = \{3, 4, 5, 6, 7, 8, 9, 10, 56, 57, 58, 59\}$, $X_3 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79\}$, $X_4 = \{29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$.

Coloring classes of Table 26: $X_1 = \{1, 2, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59\}$, $X_2 = \{3, 4, 5, 6, 7, 8, 9, 10\}$, $X_3 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79\}$, $X_4 = \{29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$.

						BSTS(91)									
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
58 56	58 54	58 51	58 53	58 52	58 55	58 50	58 78	70 59	70 61	70 58	70 55	70 60	70 62	70 50	
59 57	59 55	59 52	59 50	59 53	59 54	59 51	59 80	71 60	71 59	71 61	71 62	71 58	71 63	71 51	
60 51	60 56	60 53	60 52	60 54	60 57	60 74	60 55	72 61	72 58	72 57	72 60	72 62	72 64	72 54	
61 50	61 57	61 55	61 54	61 56	61 52	61 53	61 82	73 62	73 63	73 67	73 59	73 61	73 65	73 55	
62 47	62 48	62 54	62 55	62 57	62 49	62 56	62 79	74 63	74 62	74 66	74 54	74 64	74 67	74 56	
63 46	63 49	63 57	63 56	63 55	63 48	63 70	63 54	75 64	75 66	75 60	75 61	75 63	75 68	75 57	
64 49	64 46	64 56	64 57	64 50	64 51	64 47	64 81	76 65	76 64	76 68	76 56	76 67	76 69	76 47	
65 48	65 47	65 50	65 51	65 46	65 56	65 75	65 57	77 66	77 69	77 63	77 67	77 65	77 46	77 48	
66 53	66 50	66 49	66 48	66 51	66 47	66 76	66 46	78 67	78 65	78 64	78 48	78 66	78 47	78 46	
67 52	67 51	67 46	67 49	67 47	67 50	67 48	67 83	79 68	79 46	79 59	79 66	79 69	79 48	79 49	
68 55	68 52	68 47	68 46	68 48	68 53	68 77	68 49	80 46	80 68	80 51	80 69	80 47	80 49	80 67	
69 54	69 53	69 48	69 47	69 49	69 46	69 84	69 52	81 47	81 48	81 54	81 68	81 46	81 55	81 69	
70 85	70 86	70 87	70 88	70 89	70 90	73 85	70 91	82 48	82 47	82 46	82 65	82 49	82 56	82 68	
84 86	85 87	86 88	87 89	88 90	89 91	72 86	71 90	83 49	83 51	83 55	83 53	83 48	83 57	83 66	
83 87	84 88	85 89	86 90	87 91	71 88	71 87	72 89	84 50	84 49	84 53	84 57	84 52	84 58	84 65	
82 88	83 89	84 90	85 91	71 86	72 87	88 91	73 88	85 51	85 56	85 69	85 58	85 50	85 53	85 64	
81 89	82 90	83 91	71 84	72 85	73 86	89 90	74 87	86 52	86 55	86 56	86 51	86 57	86 59	86 63	
80 90	81 91	71 82	72 83	73 84	74 85	80 81	75 86	87 57	87 50	87 52	87 47	87 53	87 60	87 62	
79 91	71 80	72 81	73 82	74 83	75 84	79 82	76 85	88 54	88 52	88 48	88 49	88 56	88 51	88 61	
71 78	72 79	73 80	74 81	75 82	76 83	78 83	77 84	89 56	89 57	89 49	89 52	89 54	89 50	89 60	
72 77	73 78	74 79	75 80	76 81	77 82	46 49	47 48	90 53	90 54	90 50	90 46	90 55	90 52	90 59	
73 76	74 77	75 78	76 79	77 80	78 81	52 57	50 53	91 55	91 53	91 47	91 50	91 51	91 54	91 58	
74 75	75 76	76 77	77 78	78 79	79 80	54 55	51 56	58 69	60 67	62 65	63 64	59 68	61 66	52 53	

table 22

31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
70 52	70 46	70 54	70 68	70 47	70 56	70 69	70 57	70 66	70 53	70 64	70 48	70 49	70 65	70 67
71 53	71 47	71 50	71 69	71 46	71 68	71 65	71 56	71 67	71 54	71 49	71 66	71 48	71 57	71 64
72 55	72 48	72 51	72 67	72 49	72 53	72 66	72 59	72 68	72 56	72 46	72 47	72 69	72 63	72 65
73 54	73 49	73 52	73 60	73 48	73 58	73 64	73 53	73 50	73 69	73 68	73 57	73 46	73 66	73 47
74 50	74 51	74 57	74 61	74 53	74 48	74 68	74 69	74 59	74 46	74 65	74 55	74 47	74 49	74 58
75 46	75 52	75 56	75 50	75 54	75 47	75 67	75 49	75 51	75 48	75 55	75 58	75 59	75 62	75 69
76 48	76 53	76 58	76 51	76 50	76 54	76 63	76 60	76 61	76 55	76 57	76 49	76 62	76 59	76 46
77 47	77 55	77 59	77 57	77 52	77 61	77 62	77 64	77 58	77 50	77 51	77 56	77 60	77 54	77 49
78 49	78 56	78 62	78 53	78 55	78 63	78 61	78 68	78 52	78 57	78 54	78 59	78 51	78 69	78 60
79 51	79 57	79 60	79 55	79 56	79 64	79 58	79 65	79 63	79 47	79 52	79 50	79 53	79 67	79 61
80 66	80 60	80 61	80 54	80 58	80 65	80 57	80 48	80 62	80 64	80 53	80 52	80 55	80 50	80 63
81 67	81 61	81 63	81 62	81 59	81 57	81 52	81 51	81 56	81 49	81 60	81 65	81 66	81 58	81 50
82 69	82 66	82 67	82 58	82 60	82 59	82 51	82 50	82 55	82 62	82 63	82 53	82 52	82 64	82 57
83 68	83 63	83 64	83 65	83 61	83 46	83 60	83 47	83 54	83 58	83 62	83 69	83 56	83 52	83 59
84 64	84 67	84 66	84 63	84 62	84 60	84 59	84 55	84 48	84 68	84 61	84 46	84 54	84 47	84 56
85 65	85 62	85 68	85 66	85 63	85 67	85 55	85 46	85 49	85 59	85 47	85 60	85 61	85 48	85 54
86 62	86 65	86 69	86 64	86 67	86 49	86 54	86 66	86 47	86 60	86 48	86 61	86 58	86 46	86 68
87 63	87 64	87 65	87 59	87 69	87 51	87 49	87 58	87 46	87 61	87 66	87 68	87 67	87 56	87 48
88 60	88 58	88 47	88 46	88 64	88 66	88 50	88 67	88 69	88 65	88 59	88 62	88 63	88 68	88 53
89 61	89 59	89 46	89 47	89 65	89 69	89 48	89 62	89 64	89 63	89 58	89 67	89 68	89 53	89 66
90 58	90 69	90 48	90 49	90 68	90 62	90 47	90 61	90 65	90 66	90 67	90 63	90 64	90 60	90 51
91 59	91 68	91 49	91 48	91 66	91 52	91 46	91 63	91 60	91 67	91 69	91 64	91 65	91 61	91 62
56 57	50 54	53 55	52 56	51 57	50 55	53 56	52 54	53 57	51 52	50 56	51 54	50 57	51 55	52 55

table 22

BSTS(99)																
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
62 67	62 68	62 72	62 73	62 75	50 62	50 69	50 88	50 90	62 51	62 52	62 53	62 54	62 55	62 99	62 57	62 58
63 71	63 73	63 75	63 65	63 72	51 76	51 85	51 92	51 66	63 52	64 53	63 55	63 53	63 91	63 57	63 58	63 78
64 77	64 72	64 68	64 69	64 65	52 70	52 77	52 86	52 88	64 50	65 61	64 51	64 52	64 54	64 58	64 80	64 57
65 74	65 67	65 76	66 68	66 69	53 74	53 93	53 97	53 73	65 60	66 50	65 52	65 58	65 57	65 59	65 84	65 50
66 75	66 71	66 74	67 72	67 77	54 77	54 95	54 63	54 98	66 53	67 55	66 61	66 60	66 59	66 52	66 55	66 54
68 70	70 76	67 71	71 77	68 74	55 97	55 75	55 96	55 64	67 54	68 60	67 60	67 61	67 94	67 53	67 52	67 56
69 73	69 77	69 70	75 76	73 76	56 95	56 97	56 68	56 74	68 61	69 59	68 59	68 50	68 51	68 54	68 53	68 52
72 76	74 75	73 77	70 74	70 71	57 78	57 74	57 71	57 93	69 55	70 57	69 58	69 83	69 60	69 61	69 54	69 51
78 84	78 86	78 87	78 88	78 91	58 72	58 78	58 74	58 92	70 58	71 89	70 56	70 55	70 95	70 88	70 51	70 91
79 98	79 80	79 94	79 90	79 99	59 92	59 73	59 93	59 76	71 56	72 54	71 54	71 59	71 50	71 55	71 60	71 61
80 91	81 95	80 98	80 95	80 88	60 88	60 90	60 76	60 77	72 59	73 56	72 57	72 56	72 96	72 50	72 61	72 60
81 93	82 92	81 91	81 89	85 93	61 84	61 76	61 77	61 87	73 57	74 80	73 50	73 51	73 52	73 97	73 93	73 90
82 96	83 90	82 97	82 86	82 94	80 99	79 81	78 79	78 99	74 91	75 58	74 81	74 90	74 61	74 51	74 50	74 84
83 85	84 94	83 89	83 94	83 95	81 98	80 84	80 82	79 83	75 87	76 86	75 89	75 57	75 53	75 60	75 56	75 59
86 90	85 96	84 93	87 93	86 98	83 96	82 99	81 83	80 97	76 81	77 92	76 92	76 99	76 58	76 56	76 94	76 53
87 97	87 99	85 99	84 92	89 96	85 94	83 98	84 95	81 96	77 82	78 81	77 78	77 98	77 56	77 79	77 59	77 55
88 94	88 91	90 92	85 98	84 90	86 93	86 88	85 89	82 95	78 80	79 87	84 99	78 97	78 98	78 94	78 96	79 89
89 92	89 93	86 96	91 99	87 92	89 90	87 89	87 94	84 91	79 85	82 93	83 87	79 84	93 99	80 81	79 86	80 83
95 99	97 98	88 95	96 97	81 97	79 82	91 94	90 91	85 86	83 88	83 99	82 88	80 85	79 97	82 83	81 85	81 92
50 51	50 52	50 53	50 54	50 55	87 91	92 96	98 99	89 94	84 86	84 98	80 90	81 82	80 86	84 89	82 90	82 85
52 61	51 53	52 54	53 55	54 56	63 64	62 65	62 70	62 71	89 99	85 88	79 91	86 87	82 89	85 91	83 91	94 97
53 60	54 61	51 55	52 56	53 57	65 73	63 67	64 67	63 70	90 98	90 96	93 97	88 89	81 84	86 95	87 95	86 99
54 59	55 60	56 61	51 57	52 58	66 67	64 71	65 69	65 75	92 94	94 95	86 94	91 95	87 88	87 96	88 98	87 98
55 58	56 59	57 60	58 61	51 59	68 75	66 70	66 73	67 68	93 96	91 97	85 95	92 93	85 90	90 93	89 97	88 93
56 57	57 58	58 59	59 60	60 61	69 71	68 72	72 75	69 72	95 97	63 51	96 98	94 96	83 92	92 98	92 99	95 96

table 23

18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
62 98	62 60	62 61	62 59	62 79	62 56	62 80	62 82	62 84	62 89	62 91	62 94	78 60	78 62	78 64	78 65
63 60	63 61	63 50	63 56	63 59	63 82	63 79	63 83	63 86	63 88	63 97	63 87	79 64	79 76	79 66	79 71
64 96	64 56	64 99	64 61	64 60	64 59	64 95	64 92	64 90	64 83	64 81	64 85	80 70	80 65	80 63	80 72
65 54	65 55	65 53	65 79	65 56	65 51	65 81	65 85	65 91	65 94	65 98	65 99	81 62	81 63	81 67	81 66
66 56	66 85	66 58	66 80	66 57	66 87	66 83	66 84	66 98	66 96	66 92	66 95	82 75	82 64	82 65	82 60
67 57	67 50	67 59	67 58	67 51	67 89	67 99	67 98	67 95	67 80	67 79	67 84	83 50	83 74	83 52	83 53
68 55	68 93	68 96	68 86	68 58	68 57	68 94	68 88	68 79	68 82	68 83	68 80	84 54	84 50	84 51	84 52
69 52	69 53	69 56	69 57	69 85	69 91	69 82	69 80	69 81	69 97	69 90	69 88	85 53	85 52	85 60	85 59
70 50	70 59	70 54	70 53	70 61	70 60	70 84	70 81	70 93	70 99	70 82	70 83	86 56	86 59	86 50	86 51
71 53	71 51	71 52	71 93	71 86	71 58	71 97	71 91	71 87	71 84	71 85	71 81	87 59	87 67	87 73	87 64
72 51	72 52	72 94	72 55	72 89	72 53	72 87	72 79	72 99	72 85	72 78	72 82	88 55	88 51	88 53	88 61
73 58	73 54	73 95	73 60	73 55	73 61	73 98	73 96	73 82	73 81	73 80	73 79	89 61	89 66	89 70	89 73
74 59	74 98	74 60	74 52	74 54	74 55	74 88	74 87	74 89	74 78	74 86	74 96	90 58	90 53	90 56	90 54
75 61	75 95	75 51	75 50	75 52	75 54	75 90	75 99	75 78	75 86	75 94	75 92	91 73	91 55	91 57	91 56
76 93	76 57	76 55	76 54	76 50	76 52	76 89	76 78	76 83	76 87	76 96	76 91	92 67	92 60	92 54	92 55
77 83	77 58	77 57	77 51	77 53	77 50	77 85	77 86	77 94	77 90	77 89	77 97	93 77	93 61	93 69	93 62
78 89	78 92	78 83	78 95	78 82	78 85	78 93	89 95	80 96	79 92	84 87	78 90	94 57	94 56	94 55	94 58
79 95	79 96	79 93	85 87	81 87	79 88	86 92	90 97	88 97	91 93	88 99	86 89	95 65	95 69	95 59	95 68
80 94	80 89	80 87	90 94	80 92	80 93	91 96	93 94	85 92	95 98	93 95	93 98	96 52	96 54	96 58	96 57
81 88	81 94	81 86	89 91	83 93	81 90	50 56	50 57	50 58	50 59	50 60	50 61	97 51	97 58	97 61	97 50
82 87	82 91	82 84	84 96	84 97	83 84	55 57	56 58	57 59	58 60	59 61	51 60	98 71	98 68	98 72	98 63
84 85	83 86	85 97	83 97	88 96	86 97	54 58	55 59	56 60	57 61	51 58	52 59	99 68	99 57	99 77	99 74
86 91	84 88	88 90	82 98	90 95	92 95	53 59	54 60	55 61	51 56	52 57	53 58	63 66	70 72	68 76	67 70
90 99	87 90	91 92	81 99	94 99	94 98	52 60	53 61	51 54	52 55	53 56	54 57	69 76	71 73	71 75	69 75
92 97	97 99	89 98	88 92	91 98	96 99	51 61	51 52	52 53	53 54	54 55	55 56	72 74	75 77	62 74	76 77

table 23

34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49
78 66	78 59	78 68	78 69	78 70	78 71	78 73	78 50	78 51	78 52	78 61	78 54	78 55	78 67	78 56	78 53
79 75	79 51	79 52	79 53	79 54	79 55	79 56	79 57	79 58	79 59	79 60	79 61	79 74	79 69	79 50	79 70
80 71	80 76	80 77	80 50	80 51	80 52	80 53	80 54	80 55	80 56	80 57	80 58	80 59	80 60	80 61	80 75
81 68	81 72	81 75	81 61	81 50	81 51	81 52	81 77	81 54	81 55	81 53	81 57	81 58	81 59	81 60	81 56
82 51	82 66	82 67	82 71	82 52	82 53	82 54	82 55	82 61	82 76	82 56	82 50	82 57	82 58	82 74	82 59
83 54	83 55	83 56	83 57	83 58	83 62	83 59	83 60	83 72	83 61	83 73	83 75	83 67	83 51	83 65	83 71
84 53	84 60	84 55	84 59	84 75	84 58	84 57	84 56	84 63	84 77	84 69	84 76	84 68	84 64	84 73	84 72
85 58	85 57	85 54	85 56	85 55	85 63	85 50	85 61	85 76	85 67	85 68	85 62	85 70	85 73	85 75	85 74
86 61	86 69	86 53	86 66	86 67	86 73	86 70	86 72	86 60	86 62	86 54	86 55	86 65	86 57	86 58	86 64
87 65	87 68	87 70	87 58	87 53	87 50	87 77	87 52	87 57	87 54	87 62	87 69	87 56	87 55	87 51	87 60
88 59	88 58	88 57	88 76	88 56	88 77	88 75	88 62	88 64	88 65	88 67	88 66	88 71	88 54	88 72	88 73
89 64	89 65	89 63	89 68	89 57	89 69	89 58	89 59	89 56	89 60	89 50	89 51	89 53	89 52	89 55	89 54
90 55	90 52	90 51	90 63	90 62	90 68	90 66	90 65	90 67	90 57	90 70	90 71	90 72	90 61	90 59	90 76
91 52	91 50	91 60	91 54	91 77	91 59	91 64	91 51	91 66	91 53	91 75	91 68	91 61	91 72	91 67	91 58
92 56	92 61	92 50	92 62	92 63	92 57	92 65	92 69	92 52	92 70	92 71	92 72	92 73	92 74	92 53	92 68
93 63	93 56	93 64	93 65	93 66	93 67	93 72	93 74	93 75	93 58	93 51	93 60	93 52	93 50	93 54	93 55
94 60	94 54	94 59	94 52	94 73	94 74	94 61	94 71	94 53	94 51	94 66	94 70	94 64	94 63	94 69	94 50
95 62	95 63	95 71	95 72	95 74	95 76	95 51	95 58	95 77	95 50	95 55	95 52	95 60	95 53	95 57	95 61
96 50	96 53	96 61	96 51	96 60	96 56	96 62	96 63	96 65	96 69	96 59	96 67	96 75	96 71	96 70	96 77
97 57	97 62	97 65	97 64	97 72	97 60	97 67	97 68	97 70	97 74	97 76	97 59	97 54	97 75	97 66	97 52
98 76	98 64	98 69	98 55	98 59	98 61	98 60	98 53	98 50	98 75	98 58	98 56	98 51	98 70	98 52	98 57
99 69	99 71	99 58	99 60	99 61	99 54	99 55	99 66	99 59	99 73	99 52	99 53	99 50	99 56	99 63	99 51
67 74	67 73	62 66	67 75	64 76	64 75	63 76	64 70	62 69	63 68	63 77	63 74	62 76	62 77	62 64	62 63
70 73	70 75	72 73	70 77	65 71	65 70	68 71	67 76	68 73	64 66	64 74	65 77	63 69	65 68	68 77	65 66
72 77	74 77	74 76	73 74	68 69	66 72	69 74	73 75	71 74	71 72	65 72	64 73	66 77	66 76	71 76	67 69

table 23

BSTS(99)																																	
1		2		3		4		5		6		7		8		9		10		11		12		13		14		15		16		17	
50	54	50	55	50	56	50	57	50	58	59	50	60	50	61	50	62	50	63	50	53	50	64	50	65	50	66	50	67	50	67	52	69	50
51	53	51	54	51	55	51	56	51	57	58	51	59	51	60	51	61	51	62	51	63	51	70	52	74	51	73	51	72	51	68	50	70	51
52	56	52	57	52	58	52	59	52	60	61	52	62	52	63	52	53	52	54	52	55	52	75	51	69	52	68	52	66	52	71	51	65	52
55	57	53	58	54	53	63	55	54	59	53	68	53	64	53	66	54	64	53	65	54	75	65	80	64	55	64	60	64	63	64	62	64	58
58	59	56	59	60	57	61	53	53	63	54	72	54	71	54	67	55	73	55	72	56	64	66	96	66	59	65	59	65	77	65	60	66	61
60	61	60	63	61	63	62	60	62	56	55	67	55	68	55	70	56	65	56	66	57	65	67	62	67	79	67	53	68	56	66	54	67	63
62	63	61	62	59	62	54	58	61	55	56	73	56	74	56	71	57	70	57	71	58	74	68	61	68	58	69	54	69	57	69	63	68	82
64	65	64	66	64	67	64	68	64	69	57	66	57	67	57	74	58	66	58	67	59	67	69	58	70	62	70	63	70	84	70	61	71	83
66	75	67	65	66	68	67	69	68	70	60	74	58	75	58	73	59	69	59	70	60	66	71	55	71	60	71	62	71	58	72	94	72	56
67	74	68	75	65	69	66	70	67	71	62	65	61	65	59	64	60	67	60	68	61	72	72	59	72	57	72	81	73	60	73	92	73	57
68	73	69	74	70	75	65	71	66	72	63	75	63	66	62	72	63	68	61	69	62	73	73	54	73	61	74	78	74	61	74	55	74	54
69	72	70	73	71	74	72	75	65	73	64	70	70	72	69	75	72	74	64	74	68	71	74	53	75	99	75	56	75	59	75	53	75	62
71	70	72	71	72	73	74	73	75	74	69	71	69	73	65	68	71	75	73	75	69	70	56	60	53	56	55	58	53	62	56	58	53	59
76	77	76	78	76	79	76	80	76	81	76	82	76	83	76	84	76	85	76	86	76	87	57	63	54	63	57	61	54	55	59	57	55	60
78	99	79	77	78	80	79	81	80	82	81	83	82	84	83	85	84	86	85	87	86	88	76	88	76	89	76	91	76	92	76	93	76	94
79	98	80	99	77	81	78	82	79	83	80	84	81	85	82	86	83	87	84	88	85	89	87	89	88	90	90	92	91	93	91	95	93	95
80	97	81	98	82	99	77	83	78	84	79	85	80	86	81	87	82	88	83	89	84	90	86	90	87	91	89	93	90	94	90	96	92	96
81	96	82	97	83	98	84	99	77	85	78	86	79	87	80	88	81	89	82	90	83	91	85	91	86	92	88	94	89	95	89	97	91	97
82	95	83	96	84	97	85	98	86	99	87	77	88	78	89	79	90	80	81	91	82	92	84	92	85	93	87	95	88	96	88	98	90	98
83	94	84	95	85	96	86	97	87	98	88	99	77	89	78	90	79	91	80	92	81	93	93	83	94	84	86	96	87	97	87	99	89	99
84	93	85	94	86	95	87	96	88	97	89	98	90	99	77	91	78	92	79	93	80	94	82	94	83	95	85	97	86	98	77	86	77	88
85	92	86	93	87	94	88	95	89	96	90	97	91	98	92	99	77	93	78	94	79	95	81	95	82	96	84	98	85	99	85	78	78	87
86	91	87	92	88	93	89	94	90	95	91	96	92	97	93	98	94	99	77	95	78	96	79	97	81	97	83	99	78	83	79	84	79	86
87	90	88	91	89	92	90	93	91	94	92	95	93	96	94	97	95	98	96	99	77	97	78	98	80	98	77	82	79	82	80	83	80	85
88	89	89	90	90	91	91	92	92	93	93	94	94	95	95	96	96	97	97	98	98	99	77	99	77	78	79	80	80	81	81	82	81	84

table 24

18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
70 50	71 50	72 50	73 50	74 50	50 51	50 52	75 50	77 50	76 50	76 52	77 52	77 56	76 54	77 61	76 60
69 51	68 51	67 51	66 51	65 51	71 52	64 51	64 76	79 52	79 54	77 54	78 54	78 52	78 63	80 54	78 59
64 52	75 52	74 52	72 52	73 52	64 90	65 90	65 99	82 54	80 52	78 50	79 50	80 50	80 62	81 52	79 61
65 54	64 57	64 85	64 61	64 91	65 83	66 85	66 98	83 56	82 60	81 60	81 63	81 54	81 50	83 50	82 50
66 80	65 55	65 63	65 58	66 62	66 92	67 86	67 77	85 58	84 56	82 62	84 62	83 63	83 52	84 57	84 52
67 56	66 88	66 55	67 94	67 89	67 95	68 93	68 57	86 63	86 58	84 63	87 61	85 62	85 60	85 56	86 54
68 87	67 61	68 62	68 59	68 54	68 84	69 95	69 97	89 61	87 62	85 61	88 56	86 61	87 59	88 63	87 57
71 61	69 62	69 60	69 53	69 55	69 56	70 76	70 58	91 62	89 63	88 58	89 58	87 60	88 61	90 62	89 62
72 53	70 53	70 54	70 60	70 56	70 96	71 88	71 78	92 60	91 61	90 59	90 57	89 59	89 56	91 60	90 63
73 63	72 58	71 59	71 63	71 53	72 89	72 97	72 60	94 59	93 57	91 57	91 59	91 58	92 53	93 55	91 53
74 62	73 81	73 53	74 79	72 63	73 91	73 87	73 59	95 57	95 55	92 55	93 53	93 51	93 58	95 51	92 51
75 60	74 59	75 86	75 55	75 57	74 82	74 98	74 63	96 55	96 59	96 56	94 51	94 53	96 51	96 53	94 55
57 58	56 63	58 61	54 56	58 60	75 93	75 61	56 61	97 53	98 53	97 51	97 60	97 57	97 55	99 58	96 58
55 59	54 60	56 57	57 62	59 61	53 57	53 60	53 55	98 51	99 51	99 53	98 55	99 55	99 57	97 59	97 56
76 95	76 96	76 97	76 98	76 90	54 61	54 57	54 62	65 76	65 78	65 93	65 85	64 98	64 77	64 78	64 99
94 96	95 97	96 98	97 99	88 92	55 62	55 56	79 96	66 87	66 81	66 95	66 76	67 76	66 86	66 89	66 77
93 97	94 98	95 99	96 77	87 93	58 63	58 62	80 95	67 99	67 88	67 98	67 96	68 88	67 91	67 82	67 83
92 98	93 99	77 94	78 95	86 94	59 60	59 63	81 94	68 81	68 97	68 86	68 92	69 84	68 95	68 98	68 80
91 99	77 92	78 93	80 93	85 95	76 99	82 83	82 93	69 90	69 94	69 79	70 83	70 79	69 98	69 76	69 88
77 90	78 91	79 92	81 92	84 96	77 98	92 94	83 92	70 93	70 77	70 87	71 80	71 82	71 79	70 86	70 81
78 89	79 90	80 91	82 91	83 97	97 78	77 84	84 91	72 78	71 90	71 94	72 95	72 92	72 82	71 92	71 85
79 88	80 89	81 90	83 90	82 98	79 94	78 81	85 90	73 84	73 83	72 80	73 82	73 95	73 90	73 79	72 93
81 86	82 87	82 89	84 89	81 99	85 86	89 91	86 89	74 80	74 92	74 83	74 86	74 96	74 84	74 94	73 98
82 85	83 86	83 88	85 88	77 80	81 88	79 99	87 88	75 88	75 85	75 89	69 99	75 90	75 94	75 87	75 95
83 84	84 85	84 87	86 87	78 79	80 87	80 96	51 52	64 71	64 72	64 73	64 75	65 66	65 70	65 72	65 74

table 24

34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49
77 62	78 62	76 58	76 59	76 63	76 61	76 62	77 57	76 56	76 57	77 59	76 55	77 55	78 51	76 51	76 53
80 60	79 56	77 60	78 56	78 57	79 63	81 57	78 60	77 63	77 58	78 55	77 53	79 53	79 55	78 53	77 51
81 59	80 63	78 61	79 60	79 59	80 58	82 59	80 56	82 58	81 56	81 51	79 51	80 51	80 53	79 58	78 58
83 54	81 61	79 57	80 61	81 53	82 57	83 53	81 58	83 51	82 51	82 53	83 60	85 63	82 56	80 57	79 62
85 50	83 55	80 55	81 62	84 50	84 54	84 51	83 62	84 61	83 59	83 61	84 58	86 59	83 57	82 63	80 59
86 52	85 52	82 52	82 55	88 51	85 53	85 54	84 53	85 59	85 55	88 62	85 57	88 57	84 59	84 55	81 55
87 58	86 50	85 51	83 58	90 52	86 51	86 60	86 55	86 57	87 53	89 57	86 62	90 56	87 54	87 56	82 61
88 55	89 51	87 50	86 53	91 55	89 50	88 52	87 51	87 55	88 60	90 58	88 59	92 58	89 60	90 60	84 60
90 51	90 53	89 53	87 52	93 56	90 55	89 55	88 54	88 53	89 52	91 56	91 63	93 54	90 61	91 54	86 56
92 56	92 54	90 54	88 50	94 62	92 52	91 50	90 50	93 50	93 63	92 50	93 52	94 61	92 62	92 59	87 63
93 61	93 60	92 63	91 51	95 60	93 59	92 61	91 52	94 52	95 50	94 60	94 50	95 62	94 63	93 62	89 54
95 53	94 57	95 59	92 57	97 61	94 56	94 58	95 61	95 54	96 54	96 52	95 56	97 50	95 58	95 52	96 57
96 63	97 58	96 62	94 54	98 58	96 60	98 56	97 63	97 62	98 61	97 54	98 54	98 60	96 50	96 61	97 52
98 57	99 59	99 56	95 63	99 54	98 62	99 63	98 59	99 60	99 62	98 63	99 61	99 52	98 52	98 50	99 50
64 97	64 87	64 88	64 93	64 83	64 95	64 79	64 94	64 96	64 84	64 80	64 82	64 81	64 86	64 89	64 92
66 84	65 84	65 98	65 89	65 96	65 81	65 97	65 79	65 92	65 94	65 95	65 87	65 82	65 88	65 86	65 91
67 78	68 77	67 93	67 97	67 80	67 87	66 78	66 82	66 79	66 90	66 93	66 97	66 83	66 91	66 99	66 94
68 76	69 96	68 83	68 90	68 89	68 99	69 80	68 96	68 91	68 78	68 79	67 92	67 84	67 81	67 85	67 90
69 82	70 82	70 94	69 77	69 86	69 78	70 95	69 89	69 81	69 92	69 87	70 90	69 91	69 85	68 94	68 85
70 91	71 95	71 84	70 98	70 92	70 97	71 87	71 93	70 89	70 80	71 86	71 89	70 78	70 99	69 83	69 93
71 99	72 98	72 91	72 99	71 77	71 91	72 90	72 76	71 98	71 97	74 76	73 80	71 96	71 76	71 81	70 88
72 79	73 76	73 86	73 96	72 85	72 77	73 77	73 85	73 78	72 86	73 99	74 81	72 87	73 93	72 88	72 83
73 94	74 88	74 97	74 85	74 87	73 88	74 93	74 99	74 90	74 91	72 84	75 78	73 89	74 77	73 97	74 95
74 89	75 91	75 81	75 84	75 82	75 83	75 96	75 92	75 80	75 79	70 85	72 96	75 76	75 97	75 77	75 98
65 75	66 67	66 69	66 71	66 73	66 74	67 68	67 70	67 72	67 73	67 75	68 69	68 74	68 72	70 74	71 73

table 24

BSTS(99)																	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	
50 60	50 69	56 50	56 52	56 51	56 53	56 55	56 54	56 60	56 61	60 52	60 53	60 87	60 90	60 88	60 93	60 89	
51 61	51 74	57 51	57 50	57 54	57 52	57 53	57 55	57 69	57 79	61 53	61 52	61 97	61 91	61 87	61 92	61 99	
52 62	52 70	58 53	58 54	58 55	58 50	58 51	58 52	58 68	58 62	62 86	62 87	62 53	62 89	62 98	62 95	62 59	
53 79	53 73	59 52	59 55	59 53	59 54	59 50	59 51	59 70	59 78	63 85	63 94	63 52	63 53	63 99	63 59	63 58	
54 64	54 72	54 55	51 53	50 52	51 55	52 54	50 53	50 51	50 54	64 96	64 95	64 86	64 52	64 59	64 58	64 81	
55 77	55 71	60 63	60 64	60 65	60 66	60 67	60 68	52 55	53 55	65 97	65 88	65 98	65 59	65 58	65 52	65 54	
65 76	68 75	62 64	63 65	64 66	65 67	66 68	67 69	53 54	51 52	66 84	66 93	66 59	66 58	66 52	66 55	66 87	
66 75	67 76	61 65	62 66	63 67	64 68	65 69	66 70	67 71	60 70	67 98	67 59	67 58	67 55	67 53	67 89	67 52	
67 74	66 77	66 79	61 67	62 68	63 69	64 70	65 71	66 72	69 71	68 59	68 51	68 99	68 94	68 55	68 80	68 53	
68 73	65 78	67 78	68 79	61 69	62 70	63 71	64 72	65 73	68 72	69 82	69 58	69 85	69 88	69 54	69 53	69 55	
69 72	64 79	68 77	69 78	70 79	71 61	72 62	63 73	64 74	67 73	70 57	70 50	70 51	70 96	70 83	70 54	70 82	
70 71	61 63	69 76	70 77	71 78	72 79	61 73	62 74	63 75	66 74	71 56	71 57	71 50	71 51	71 81	71 94	71 88	
63 78	60 62	70 75	71 76	72 77	73 78	74 79	61 75	62 76	65 75	72 58	72 56	72 57	72 95	72 84	72 91	72 83	
80 81	80 82	71 74	72 75	73 76	74 77	75 78	76 79	61 77	64 76	73 50	73 89	73 56	73 57	73 51	73 90	73 84	
82 99	81 83	72 73	73 74	74 75	75 76	76 77	77 78	78 79	63 77	74 99	74 92	74 82	74 56	74 57	74 96	74 86	
83 98	84 99	80 89	80 90	80 83	80 84	80 85	80 86	80 87	80 88	75 83	75 91	75 81	75 50	75 56	75 57	75 85	
84 97	85 98	88 90	89 91	82 84	83 85	84 86	85 87	86 88	87 89	76 51	76 90	76 55	76 54	76 50	76 56	76 57	
85 96	86 97	87 91	88 92	81 85	82 86	83 87	84 88	85 89	86 90	77 81	77 80	77 54	77 93	77 86	77 50	77 56	
86 95	87 96	86 92	87 93	86 99	81 87	82 88	83 89	84 90	85 91	78 55	78 54	78 83	78 92	78 85	78 51	78 50	
87 94	88 95	85 93	86 94	87 98	88 99	81 89	82 90	83 91	84 92	79 54	79 55	79 84	79 80	79 82	79 97	79 51	
88 93	89 94	84 94	85 95	88 97	89 98	90 99	91 81	82 92	83 93	80 91	86 96	80 92	87 97	80 93	88 98	80 94	
89 92	90 93	83 95	84 96	89 96	90 97	91 98	92 99	93 81	82 94	90 92	85 97	91 93	86 98	92 94	87 99	93 95	
90 91	91 92	82 96	83 97	90 95	91 96	92 97	93 98	94 99	81 95	89 93	84 98	90 94	85 99	91 95	81 86	92 96	
56 57	56 58	81 97	82 98	91 94	92 95	93 96	94 97	95 98	96 99	88 94	83 99	89 95	84 81	90 96	82 85	91 97	
58 59	57 59	98 99	81 99	92 93	93 94	94 95	95 96	96 97	97 98	87 95	81 82	88 96	82 83	89 97	83 84	90 98	

table 25

18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
60 59	60 58	60 57	60 96	60 55	60 94	60 84	60 80	60 85	60 54	60 51	80 50	80 51	80 67	80 70	80 52
61 58	61 59	61 50	61 95	61 83	61 55	61 93	61 98	61 54	61 80	61 57	81 66	81 72	81 65	81 60	81 67
62 93	62 81	62 56	62 57	62 51	62 80	62 55	62 54	62 94	62 99	62 50	82 77	82 50	82 62	82 51	82 56
63 95	63 90	63 51	63 50	63 57	63 96	63 54	63 55	63 86	63 81	63 56	83 57	83 60	83 50	83 74	83 54
64 53	64 89	64 80	64 56	64 50	64 57	64 51	64 97	64 55	64 98	64 82	84 63	84 57	84 77	84 50	84 62
65 80	65 53	65 95	65 51	65 56	65 50	65 57	65 99	65 93	65 55	65 91	85 79	85 73	85 57	85 71	85 50
66 94	66 88	66 53	66 80	66 54	66 56	66 50	66 57	66 51	66 82	66 83	86 56	86 58	86 51	86 59	86 55
67 96	67 82	67 94	67 54	67 92	67 51	67 56	67 50	67 57	67 97	67 90	87 52	87 67	87 58	87 55	87 68
68 52	68 83	68 96	68 97	68 84	68 98	68 85	68 56	68 50	68 57	68 54	88 61	88 52	88 78	88 57	88 51
69 97	69 52	69 93	69 94	69 91	69 95	69 92	69 51	69 56	69 59	69 84	89 59	89 68	89 52	89 75	89 65
70 55	70 84	70 97	70 98	70 85	70 99	70 86	70 81	70 58	70 56	70 53	90 64	90 69	90 56	90 68	90 61
71 92	71 87	71 52	71 53	71 90	71 54	71 91	71 96	71 59	71 58	71 89	91 55	91 74	91 79	91 52	91 66
72 51	72 86	72 55	72 93	72 53	72 94	72 90	72 59	72 87	72 50	72 52	92 58	92 55	92 64	92 76	92 57
73 91	73 54	73 98	73 52	73 86	73 81	73 59	73 58	73 92	73 96	73 55	93 54	93 71	93 55	93 56	93 59
74 90	74 50	74 54	74 55	74 52	74 59	74 58	74 95	74 88	74 53	74 85	94 65	94 59	94 61	94 58	94 64
75 54	75 55	75 92	75 99	75 59	75 58	75 52	75 82	75 53	75 51	75 88	95 78	95 70	95 59	95 73	95 53
76 98	76 85	76 99	76 59	76 58	76 52	76 87	76 53	76 91	76 83	76 86	96 62	96 56	96 66	96 53	96 58
77 57	77 51	77 91	77 58	77 87	77 53	77 88	77 83	77 89	77 52	77 59	97 76	97 75	97 53	97 54	97 63
78 56	78 57	78 58	78 81	78 89	78 82	78 53	78 52	78 90	78 84	78 87	98 53	98 54	98 63	98 69	98 78
79 50	79 56	79 59	79 92	79 88	79 93	79 89	79 94	79 52	79 95	79 58	99 51	99 53	99 54	99 72	99 79
81 88	80 95	81 90	82 91	80 97	83 92	80 98	84 93	80 99	85 94	80 96	60 71	76 66	60 72	77 67	60 73
82 87	94 96	82 89	83 90	96 98	84 91	97 99	85 92	81 98	86 93	95 97	70 72	65 77	71 73	66 78	72 74
83 86	93 97	83 88	84 89	95 99	85 90	81 96	86 91	82 97	87 92	94 98	69 73	64 78	70 74	65 79	71 75
84 85	92 98	84 87	85 88	94 81	86 89	95 82	87 90	96 83	88 91	93 99	68 74	63 79	69 75	61 64	70 76
89 99	91 99	85 86	86 87	82 93	87 88	83 94	88 89	84 95	89 90	81 92	67 75	61 62	68 76	62 63	69 77

table 25

34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49
80 73	80 56	80 59	80 71	80 58	80 63	80 74	80 53	80 57	80 54	80 55	80 69	80 76	80 78	80 72	80 75
81 50	81 61	81 74	81 68	81 79	81 56	81 52	81 69	81 53	81 55	81 54	81 58	81 57	81 59	81 51	81 76
82 57	82 68	82 72	82 54	82 52	82 71	82 61	82 58	82 59	82 73	82 76	82 53	82 55	82 63	82 65	82 60
83 58	83 64	83 55	83 59	83 53	83 69	83 73	83 67	83 63	83 65	83 79	83 62	83 56	83 51	83 52	83 71
84 76	84 51	84 75	84 64	84 61	84 58	84 56	84 71	84 54	84 67	84 52	84 65	84 59	84 55	84 53	84 74
85 51	85 65	85 52	85 56	85 55	85 54	85 58	85 59	85 62	85 66	85 61	85 64	85 72	85 53	85 67	85 77
86 54	86 50	86 78	86 67	86 75	86 68	86 53	86 65	86 61	86 71	86 60	86 52	86 79	86 66	86 57	86 69
87 69	87 57	87 70	87 50	87 59	87 65	87 51	87 64	87 79	87 53	87 56	87 63	87 74	87 54	87 54	87 73
88 59	88 58	88 50	88 53	88 56	88 67	88 76	88 70	88 55	88 72	88 62	88 68	88 73	88 58	88 63	88 64
89 74	89 66	89 57	89 51	89 50	89 70	89 54	89 55	89 56	89 69	89 63	89 61	89 53	89 77	89 76	89 72
90 70	90 62	90 53	90 65	90 57	90 50	90 59	90 54	90 52	90 51	90 75	90 66	90 58	90 67	90 55	90 79
91 56	91 59	91 51	91 70	91 78	91 53	91 62	91 68	91 50	91 57	91 64	91 54	91 60	91 56	91 58	91 63
92 52	92 63	92 77	92 62	92 60	92 66	92 50	92 72	92 51	92 68	92 53	92 70	92 54	92 52	92 59	92 65
93 75	93 53	93 58	93 63	93 51	93 64	93 57	93 73	93 76	93 70	93 50	93 67	93 78	93 76	93 74	93 68
94 53	94 52	94 54	94 57	94 73	94 51	94 77	94 50	94 60	94 74	94 78	94 55	94 75	94 57	94 56	94 70
95 77	95 54	95 71	95 58	95 76	95 55	95 60	95 66	95 75	95 50	95 51	95 56	95 52	95 65	95 68	95 67
96 55	96 69	96 76	96 52	96 54	96 72	96 79	96 57	96 78	96 59	96 77	96 51	96 50	96 73	96 75	96 61
97 71	97 55	97 56	97 66	97 72	97 52	97 78	97 51	97 74	97 58	97 57	97 59	97 77	97 50	97 50	97 62
98 72	98 79	98 60	98 55	98 74	98 59	98 75	98 56	98 77	98 52	98 58	98 57	98 51	98 64	98 71	98 66
99 60	99 67	99 73	99 69	99 77	99 57	99 55	99 52	99 58	99 56	99 59	99 50	99 71	99 75	99 66	99 78
78 68	60 74	69 79	60 76	62 71	60 77	63 72	60 78	64 73	60 79	65 74	60 75	61 70	60 61	60 69	56 59
67 79	73 75	68 61	75 77	70 63	76 78	71 64	77 79	72 65	78 61	73 66	74 76	69 62	62 79	61 79	57 58
66 61	72 76	62 67	74 78	64 69	75 79	65 70	61 76	66 71	62 77	67 72	73 77	63 68	69 74	62 78	50 55
62 65	71 77	63 66	73 79	65 68	61 74	66 69	62 75	67 70	63 76	68 71	72 78	64 67	68 70	70 73	51 54
63 64	70 78	64 65	61 72	66 67	62 73	67 68	63 74	68 69	64 75	69 70	71 79	65 66	71 72	64 77	52 53

table 25

BSTS(99)																	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	
50	60	50	61	60	77	60	67	60	72	60	71	60	74	60	75	60	76
51	61	51	62	61	64	61	68	61	69	61	66	61	72	61	74	61	70
52	62	52	60	62	79	62	74	62	76	62	78	62	68	62	65	62	75
53	63	53	64	63	71	63	69	63	75	63	72	63	79	63	78	63	74
54	64	54	67	65	73	64	79	64	78	64	73	64	76	64	70	64	77
55	65	55	69	66	74	65	75	65	70	65	77	65	69	66	67	65	72
56	66	56	65	67	75	66	78	66	79	67	76	66	70	68	76	66	71
57	67	57	68	68	72	70	73	67	73	68	69	67	77	69	77	67	79
58	68	58	63	69	78	71	76	68	77	70	74	71	78	71	72	68	78
59	69	59	66	70	76	72	77	71	74	75	79	73	75	73	79	69	73
70	71	70	75	50	51	50	52	50	53	50	54	50	55	50	56	50	57
72	75	71	73	52	59	51	53	51	55	51	57	51	59	51	52	51	54
73	74	72	74	53	58	54	59	52	54	52	56	52	58	53	59	52	53
76	79	76	77	54	57	55	58	56	59	53	55	53	57	54	58	55	59
77	78	78	79	55	56	56	57	57	58	58	59	54	56	55	57	56	58
80	82	80	84	80	85	80	86	80	88	80	89	80	90	80	91	80	92
81	83	81	93	81	98	81	99	81	95	81	94	81	92	81	85	81	91
94	99	82	90	82	94	82	92	82	99	82	98	82	86	82	87	82	93
84	92	83	97	83	95	83	88	83	93	83	92	83	91	83	99	83	90
85	97	94	96	84	90	84	91	84	96	84	95	84	87	84	93	84	94
86	93	85	89	86	87	85	93	85	87	85	90	85	99	86	97	85	86
87	98	86	98	88	93	87	96	86	90	86	91	88	97	88	92	87	99
95	96	87	91	89	92	89	98	89	97	87	93	89	94	89	95	88	98
89	91	88	95	91	97	90	97	91	98	88	96	93	96	90	94	89	96
88	90	92	99	96	99	94	95	92	94	97	99	95	98	96	98	95	97

table 26

18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
60 59	60 80	60 98	60 97	60 83	60 96	60 85	60 86	60 92	60 88	60 89	80 50	80 52	80 51	80 53	80 57
61 97	61 82	61 95	61 94	61 89	61 92	61 88	61 52	61 96	61 59	61 58	81 51	81 50	81 52	81 60	81 53
62 83	62 58	62 85	62 88	62 98	62 93	62 89	62 53	62 59	62 84	62 57	82 75	82 51	82 50	82 52	82 65
63 82	63 94	63 59	63 86	63 88	63 85	63 93	63 84	63 57	63 56	63 96	83 53	83 70	83 73	83 50	83 51
64 80	64 89	64 92	64 58	64 94	64 91	64 84	64 95	64 99	64 57	64 93	84 52	84 53	84 71	84 51	84 50
65 88	65 84	65 99	65 98	65 58	65 83	65 57	65 90	65 85	65 89	65 92	85 67	85 68	85 53	85 71	85 72
66 54	66 55	66 97	66 57	66 84	66 94	66 58	66 83	66 88	66 92	66 87	86 78	86 77	86 54	86 79	86 58
67 50	67 53	67 55	67 56	67 99	67 58	67 81	67 89	67 84	67 93	67 91	87 64	87 65	87 69	87 58	87 70
68 55	68 50	68 54	68 59	68 56	68 90	68 83	68 88	68 97	68 81	68 99	88 70	88 78	88 58	88 67	88 71
69 51	69 52	69 83	69 96	69 57	69 56	69 90	69 82	69 91	69 53	69 85	89 63	89 58	89 76	89 66	89 68
70 52	70 51	70 57	70 80	70 54	70 55	70 59	70 58	70 98	70 82	70 95	90 58	90 60	90 61	90 62	90 59
71 53	71 54	71 51	71 52	71 50	71 57	71 55	71 56	71 58	71 87	71 59	91 61	91 66	91 65	91 59	91 56
72 95	72 57	72 52	72 51	72 53	72 50	72 54	72 98	72 90	72 58	72 55	92 69	92 63	92 59	92 56	92 67
73 57	73 59	73 58	73 53	73 51	73 52	73 50	73 54	73 56	73 99	73 90	93 54	93 59	93 56	93 69	93 79
74 86	74 93	74 53	74 83	74 52	74 51	74 80	74 50	74 54	74 55	74 56	94 59	94 56	94 57	94 77	94 75
75 87	75 86	75 56	75 55	75 90	75 53	75 51	75 59	75 50	75 98	75 52	95 56	95 55	95 68	95 73	95 69
76 56	76 95	76 90	76 91	76 81	76 54	76 52	76 51	76 55	76 50	76 53	96 62	96 71	96 66	96 55	96 54
77 85	77 56	77 93	77 54	77 59	77 80	77 91	77 55	77 51	77 52	77 50	97 57	97 79	97 78	97 54	97 55
78 92	78 91	78 50	78 93	78 55	78 59	78 53	78 99	78 52	78 51	78 54	98 71	98 57	98 55	98 74	98 73
79 58	79 92	79 91	79 50	79 85	79 81	79 56	79 57	79 53	79 54	79 51	99 55	99 54	99 72	99 57	99 52
81 89	83 87	80 96	81 84	82 91	82 97	82 96	80 87	80 95	80 97	80 94	60 65	61 62	60 62	61 63	60 61
84 99	85 98	81 82	82 85	80 93	84 98	86 92	81 96	81 87	83 94	81 86	66 68	64 67	63 64	64 65	62 63
90 93	88 99	84 86	87 92	86 96	86 89	87 95	85 94	82 89	85 96	82 88	72 73	69 72	67 70	68 70	64 66
91 96	81 97	87 89	89 99	87 97	87 88	94 97	91 93	83 86	86 95	83 84	74 76	73 76	74 79	75 78	74 77
94 98	90 96	88 94	90 95	92 95	95 99	98 99	92 97	93 94	90 91	97 98	77 79	74 75	75 77	72 76	76 78

table 26

34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49
80 58	80 59	80 66	80 65	80 67	80 68	80 54	80 71	80 73	80 75	80 78	80 76	80 56	80 62	80 55	80 79
81 57	81 58	81 59	81 61	81 63	81 64	81 65	81 69	81 70	81 72	81 74	81 56	81 77	81 55	81 54	81 78
82 53	82 57	82 58	82 77	82 78	82 79	82 59	82 60	82 62	82 64	82 76	82 55	82 67	82 54	82 56	82 68
83 52	83 75	83 77	83 76	83 79	83 78	83 61	83 59	83 57	83 63	83 55	83 64	83 54	83 56	83 58	83 67
84 73	84 74	84 75	84 78	84 76	84 60	84 57	84 79	84 59	84 61	84 56	84 54	84 55	84 58	84 77	84 69
85 50	85 51	85 52	85 73	85 74	85 75	85 64	85 57	85 56	85 55	85 54	85 59	85 58	85 61	85 66	85 70
86 51	86 50	86 53	86 52	86 61	86 57	86 62	86 64	86 55	86 56	86 65	86 66	86 59	86 68	86 69	86 71
87 72	87 52	87 50	87 51	87 53	87 73	87 55	87 56	87 54	87 60	87 61	87 62	87 63	87 59	87 57	87 76
88 74	88 53	88 55	88 50	88 51	88 52	88 56	88 54	88 72	88 73	88 75	88 69	88 79	88 57	88 59	88 64
89 59	89 56	89 51	89 53	89 50	89 54	89 52	89 55	89 69	89 70	89 71	89 72	89 57	89 73	89 74	89 75
90 56	90 63	90 64	90 55	90 54	90 50	90 51	90 52	90 53	90 78	90 67	90 57	90 66	90 70	90 71	90 74
91 75	91 73	91 71	91 54	91 55	91 51	91 50	91 53	91 52	91 62	91 57	91 58	91 60	91 63	91 70	91 72
92 70	92 71	92 54	92 72	92 57	92 55	92 53	92 50	92 51	92 52	92 58	92 73	92 74	92 76	92 62	92 77
93 76	93 55	93 72	93 57	93 75	93 53	93 60	93 58	93 50	93 51	93 52	93 61	93 65	93 66	93 68	93 73
94 55	94 54	94 74	94 68	94 52	94 70	94 67	94 51	94 58	94 50	94 53	94 79	94 62	94 65	94 72	94 60
95 54	95 78	95 57	95 74	95 77	95 59	95 58	95 63	95 66	95 53	95 50	95 51	95 52	95 60	95 67	95 62
96 68	96 72	96 56	96 75	96 59	96 58	96 79	96 78	96 76	96 57	96 77	96 50	96 51	96 52	96 53	96 65
97 77	97 76	97 70	97 56	97 58	97 72	97 74	97 65	97 71	97 59	97 64	97 53	97 50	97 51	97 52	97 63
98 78	98 77	98 79	98 58	98 56	98 76	98 68	98 67	98 64	98 54	98 59	98 52	98 53	98 50	98 51	98 66
99 63	99 79	99 69	99 59	99 70	99 56	99 76	99 74	99 60	99 58	99 51	99 77	99 75	99 53	99 50	99 61
60 66	60 68	60 78	60 79	60 73	61 77	63 70	61 73	61 75	65 76	60 70	60 63	61 71	64 74	60 64	50 59
62 64	65 70	62 67	62 66	62 71	62 69	66 75	62 72	63 77	66 77	63 68	65 67	64 68	67 78	61 79	51 58
61 65	62 66	63 65	63 67	64 72	63 66	69 71	70 77	65 79	67 71	66 72	68 74	69 76	69 75	63 73	52 57
67 69	61 67	68 73	64 71	65 68	65 71	72 78	66 76	67 68	68 79	62 73	70 78	70 72	71 77	65 78	53 56
71 79	64 69	61 76	69 70	66 69	67 74	73 77	68 75	74 78	69 74	69 79	71 75	73 78	72 79	75 76	54 55

table 26

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